ADVANCED DYNAMICS

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PREFACE

Modern developments in various fields of applied science have greatly increased the importance of mechanics as a fundamental engineering subject. The increasing use of high-speed machines has given rise to many complex problems the solution of which requires a thorough knowledge of vibration theory and balancing. Present research with guided missiles has likewise aroused new interest in the science of ballistics and the theory of the gyroscope. Experimental studies on the behavior of structures under various kinds of dynamic loading have called attention to many new problems of structural dynamics. To meet these increasing needs for more extensive training in dynamics, many of our engineering schools now offer a number of advanced courses in this branch of mechanics. This book is intended primarily as a textbook for such courses. It may also be of interest to research and practicing engineers in dealing with dynamical problems.

In the preparation of the book, the authors have tried to present the general principles of dynamics together with their application to some of the more important technical problems encountered in various fields of engineering.

The first chapter deals with the dynamics of a particle and the solution of the differential equations of motion for various particular cases. Applications include a section on rectilinear motion with various kinds of resistance, several sections on vibration problems, one on planetary motion, and one on exterior ballistics. Since it is not always possible to obtain rigorous solutions for differential equations of motion, applications of several approximate methods of graphical and numerical integration are also shown. In general, the accuracy of such methods is entirely satisfactory for the solution of practical problems, and it has seemed desirable to give engineering students some ideas of approximate solutions of dynamics problems.

The second chapter introduces the notion of a system of particles and proceeds with a development of the important principles of linear and angular momentum and the law of conservation of energy. Applications of these principles include discussions of rocket motion, impact phenomena, and fluid motion. This chapter also includes several sections dealing with the problem of engine balancing and flywheel calculations.

These sections are developed in sufficient detail to give the student a working knowledge of this important engineering problem.

Chapter III takes up the dynamics of systems with constraints and introduces the notions of generalized coordinates and generalized forces. The method of setting up equations of motion by using the principle of virtual work in conjunction with D'Alembert's principle is treated in detail. Following this method of approach, the Lagrangian equations of motion are developed in general form and their application illustrated by numerous examples. The last section is devoted to a brief discussion of Hamilton's principle.

Chapter IV deals with the theory of small vibrations of conservative systems. Both free and forced vibrations of systems with two degrees of freedom are discussed in detail, special attention being given to the theory of dynamic dampers. Approximate methods of calculating principal frequencies of systems with several degrees of freedom are also discussed, and one section treats the lateral vibration of beams as an example of a system with an infinite number of degrees of freedom. The last two sections consider several examples of vibration about a steady state of motion and the theory of variable-speed dampers. Throughout this chapter, the Lagrangian method of writing equations of motion has been used.

The last chapter deals with the rotation of a rigid body about a fixed point, beginning with a discussion of the kinematics involved and proceeding to a development of Euler's equations of motion. As an example of the application of these equations, the theory of the gyroscope is discussed with reference to such technical uses as the gyroscopic compass, the gyroscopic pendulum, and gyroscopic stabilizers.

The book contains more material than can be covered properly in one course, and, very probably, few instructors will desire to follow it from cover to cover. In anticipation of this, each chapter has been made as nearly self-contained as possible so that the material can be used in any desired combinations. The authors have found, for example, that the material in Chap. III and portions of that in Chap. IV are adequate for a short course in Lagrangian equations and their application to vibration problems. Again, the material in Chap. I has been used in a similar way for a short course on graphical and numerical methods of integration of differential equations of dynamics. Chapters II and V might likewise be found useful for short courses dealing with engine balancing and the theory of the gyroscope.

In addition to numerous examples discussed throughout the text, the book contains approximately 150 problems without solutions. For the most part, these are attached to those sections dealing with the

development of general principles and are designed to give the student an opportunity to test his own powers of application. Answers are given to most of these problems.

Various books on both theoretical and applied dynamics were used in the preparation of this book. In this respect, acknowledgment is due to Lamb's "Dynamics," Routh's "Elementary Rigid Dynamics," and in particular to the book "Technische Dynamik" by Biezeno and Grammel for material used in the treatment of engine balancing.

Also used to some extent were the Russian books: E. L. Nikolai, "Theoretical Mechanics," vol. III, Moscow, 1939, and L. G. Lojciansky and A. J. Lurie, "Theoretical Mechanics," vol. III, Moscow, 1934.

We take this opportunity to express our thanks to Bruce Carder for assistance in the preparation of drawings and to Mrs. Evelyn Sarson for her extreme care in the typing of the manuscript.

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September, 1948

CONTENTS

NOTATIONS

A	area, amplitude
a	area, radius, acceleration
B	flexural rigidity
C	torsional rigidity
c	coefficient of viscous resistance
D,d	diameter
A,B,C,D	constants of integration
a,b,c	dimensions, coefficients, quantities
E	modulus of elasticity
e	natural logarithmic base, eccentricity
F	force
f	frequency, coefficient of friction
G	modulus of elasticity in shear
g	gravitational acceleration
H	angular momentum
h	height
I	moment of inertia
I_x,I_y,I_z	moments of inertia
I_{xy},I_{yz},I_{zx}	products of inertia
i	radius of gyration, general number, $\sqrt{-1}$
J	polar moment of inertia
j	any number
K	constant
k	spring constant
L	Lagrangian function, length
l	length
M	moment of force
m	mass of a particle
N	normal force
n	any number
O	origin of coordinates
P	force
p	angular frequency, pressure
Q	generalized force, quantity of flow
q	generalized coordinate
R	resisting force, resultant force, radius
r	radius, root
S	axial force
s	curvilinear displacement, stress

T	kinetic energy, tangential force
t	time
V	potential energy, volume, terminal velocity
v	velocity
W	weight
w	weight per unit volume
X,Y,Z	components of force
x,y,z	rectangular coordinates
$\dot{x},\dot{y},\dot{z}$	components of velocity
$\ddot{x},\ddot{y},\ddot{z}$	components of acceleration
α	phase angle
β	magnification factor
γ	mass per unit volume, damping factor
α,β,γ	angles, quantities, direction cosines
δ	deflection, variational symbol
ϵ	phase angle
λ	amplitude
μ	coefficient of friction, dynamic viscosity
ρ	mass density, radius of curvature
τ	period of vibration
ω	angular velocity
θ,ϕ,ψ	angles of rotation
ξ,η,ζ	coordinates
$\bar{F},\bar{v},\overline{AB}$	vector quantities

CHAPTER I

DYNAMICS OF A PARTICLE

1. Differential Equation of Rectilinear Motion.—A particle is defined as a material point that occupies no appreciable space. Many physical bodies are so small compared with their range of motion as to be treated as particles. *Dynamics of a particle* comprises a study of the motion of such bodies under the action of applied forces. This study is based largely on Newton's first two laws of motion. The first law states that every particle remains at rest or moves uniformly in a straight line, except in so far as it may be compelled by force to change that state. This law, sometimes called the *inertia law*, establishes the inherent resistance of matter to change in motion and indirectly defines the concept of force as the only agent that can change or tend to change the motion of a body. The second law states that when a force acts upon a given particle, that force produces an acceleration of the particle in the direction of the force and proportional to its magnitude. From this law, we see that the effect of a given force on a given particle is independent of the effect of any other force that may also act on the particle and, likewise, independent of any motion that the particle may have had before the force began to act. For example, a constant force in a given direction produces a constant acceleration in the same direction irrespective of whether the particle was initially moving in that direction or not. Thus, at the very outset, we must recognize that the motion of a particle will depend not only upon the acting forces but also upon the *initial motion.*

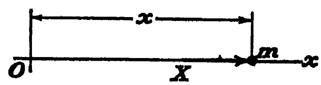


FIG. 1.

In the particular case where the resultant acting force is constant in direction and any initial motion of the particle was also in that direction, we obtain a rectilinear motion. In such case, taking the line of motion as the x-axis (Fig. 1), the acceleration of a particle will be defined by the second derivative of displacement x with respect to time t, for which we use the notation $\ddot{x}$. Then denoting the magnitude of the resultant force by X, the second law of motion may be expressed by the equation

$$m\ddot{x} = X, \tag{1}$$

where the constant of proportionality m represents the *mass* of the

particle. This is the differential equation of rectilinear motion; we see that it is an ordinary differential equation of second order.

Depending upon the law of variation of the acting force X, Eq. (1) may take many different forms. For example, we may know the acting force as a function of the independent variable time, $X = F(t)$, or as a function of the displacement of the particle along its path, $X = F(x)$. Again, in the case of resisted motion, we may have the force as a function of velocity, $X = F(\dot{x})$. Very often, the resultant force will be a combination of all of these cases, and we have

$$m\ddot{x} = F(x,\dot{x},t), \tag{1a}$$

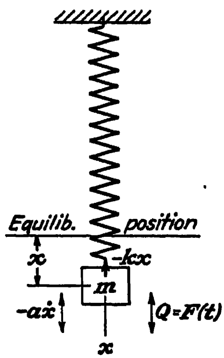


FIG. 2.

For example, if we have a spring-suspended mass m vibrating vertically in air under the action of a variable external force $Q = F(t)$ (Fig. 2), there will act, in addition to Q, a spring force proportional to the displacement x and a resisting force proportional to the velocity $\dot{x}$. Thus, Eq. (1) becomes

$$m\ddot{x} = -kx - a\dot{x} + F(t), \tag{1b}$$

and we have a *linear differential equation with constant coefficients*.[1]

In Fig. 3, we have a mass m attached to a thin cantilever spring, the effective length of which can be varied with time in any prescribed manner by moving the support A. Thus, in discussing lateral vibrations of the mass m, we have a spring characteristic that varies with time, and the equation of motion, without any external force Q, becomes

$$m\ddot{x} = -F(t)x - a\dot{x}. \tag{1c}$$

In this case, we obtain a *linear differential equation with variable coefficients*.

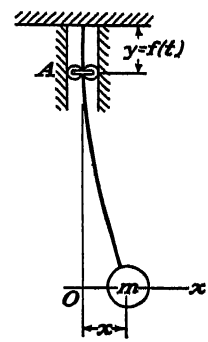


FIG. 3.

If, in Fig. 2, we take the resistance proportional to the square of the velocity and write

$$m\ddot{x} = -kx - a\dot{x}^2 + F(t), \tag{1d}$$

we obtain a *nonlinear differential equation*.

To describe completely a rectilinear motion of a particle, we need to know the displacement x for each instant of time t. Thus in any given case, our problem consists in finding a function

$$x = f(t) \tag{2}$$

[1] The solution of Eq. (1b) is discussed in Arts. 5, 6, and 7.

that satisfies Eq. (1) and also the known initial conditions of the motion. In general, a second-order differential equation will have a solution that can be represented by a function containing two arbitrary constants. In this ambiguous form, the function is called a *general solution* of the equation. It may be said to represent a family of curves in the xt-plane and any particular pair of values for the arbitrary constants defines, uniquely, one of these curves. Analytically, the problem usually resolves itself into finding the general solution and then selecting values for the two arbitrary constants in such a way as to satisfy the known *initial conditions* of the motion, namely: the initial displacement x_0 and the initial velocity $\dot{x}_0$; *i.e.*,

$$x = x_0, \qquad \dot{x} = \dot{x}_0, \qquad \text{when } t = 0. \tag{3}$$

While differentiation of a given function, to find its successive *derivatives*, can always be conducted according to definite mathematical rules, the inverse problem of integration of a given differential equation, to find its *primitive*, is more difficult, and there is no general method of procedure. In the simplest cases, where the variables can be separated, the solution can always be made by two successive steps of *simple quadrature*. Consider, for example, the case in which the acting force $X = F(t)$ (Fig. 4). Then Eq. (1) becomes

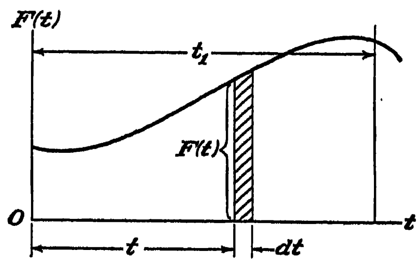


FIG. 4.

$$\ddot{x} = \frac{1}{m} F(t). \tag{a}$$

To integrate this equation, we first write

$$d\dot{x} = \frac{1}{m} F(t)dt.$$

Then, by simple quadrature, we obtain[1]

$$\dot{x} + C_1 = \frac{1}{m} \int_0^t F(t)dt,$$

where C_1 is a constant of integration. To evaluate this constant, we use the known initial condition $(\dot{x})_{t=0} = \dot{x}_0$. For $t = 0$, the integral vanishes[2]

[1] By $\int_0^t F(t)dt$ it must be understood that we mean the difference between the value of the integral function at any time t and its initial value at $t = 0$.

[2] We assume that $F(t)$ is finite at $t = 0$.

and we find $C_1 = -\dot{x}_0$. Thus the velocity-time equation becomes

$$\dot{x} = \dot{x}_0 + \frac{1}{m}\int_0^t F(t)dt. \tag{4}$$

Using the notation

$$G(t) = \int_0^t F(t)dt, \tag{b}$$

and proceeding as above with a second integration, we obtain

$$x + C_2 = \dot{x}_0 t + \frac{1}{m}\int_0^t G(t)dt.$$

Assuming now $(x)_{t=0} = x_0$, we find $C_2 = -x_0$ and the general displacement-time equation becomes

$$x = x_0 + \dot{x}_0 t + \frac{1}{m}\int_0^t G(t)dt. \tag{5}$$

For any rectilinear motion of a particle under the action of a force that can be expressed as a function of time, Eqs. (4) and (5) reduce the solution of Eq. (1) to the evaluation of two definite integrals. In connection with these equations, we note that the terms containing x_0 and $\dot{x}_0$ represent the influence of the initial motion while the integrals represent the effect of the acting force.

Example: A particle moves rectilinearly under the action of a force $X = P_0 \sin \omega t$. Find the general velocity-time and displacement-time equations.

Solution: Using Eq. (4), we obtain

$$\dot{x} = \dot{x}_0 + \frac{P_0}{\omega m}[-\cos \omega t]_0^t = \dot{x}_0 + \frac{P_0}{\omega m}(1 - \cos \omega t). \tag{c}$$

The corresponding velocity-time diagram is shown in Fig. 5*a*.

Taking

$$G(t) = \frac{P_0}{\omega}(1 - \cos \omega t),$$

and using Eq. (5), we find

$$x = x_0 + \dot{x}_0 t + \frac{P_0}{\omega^2 m}(\omega t - \sin \omega t). \tag{d}$$

The corresponding displacement-time diagram is shown in Fig. 5*b*. We see that we have a simple harmonic motion of amplitude $P_0/\omega^2 m$ and period $2\pi/\omega$ superimposed upon a uniform motion having velocity $\dot{x}_0 + P_0/\omega m$.

From Eqs. (4) and (5), we observe (1) that if no force acts upon a particle, it moves uniformly with its initial velocity $\dot{x}_0$ and (2) that if a force $F(t)$ acts only for the time dt, thereby creating a so-called impulse $F(t)dt$, it produces, in the time dt, a negligible displacement and an instantaneous change in velocity $d\dot{x} = (1/m)F(t)dt$. These two observa-

tions suggest another way of expressing the displacement-time equation for the rectilinear motion of a particle under the action of any force $X = F(t)$. Referring again to the force-time diagram in Fig. 4, we imagine that the continuous action of the force is divided into a series of infinitesimal impulses, each represented by an elemental strip of area

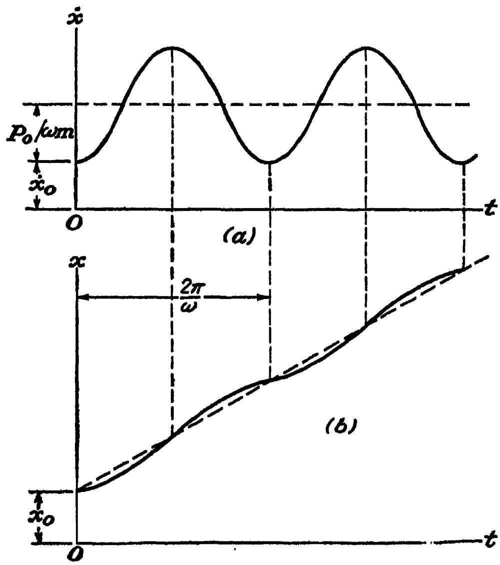


FIG. 5.

$F(t)dt$. Confining our attention to just one of these impulses, occurring at the instant t, we conclude that the displacement at any later time t_1 resulting therefrom will be

$$d\dot{x}(t_1 - t) = \frac{1}{m} F(t)dt(t_1 - t).$$

Hence, the total displacement at the time t_1, due to the succession of impulses from $t = 0$ to $t = t_1$, *i.e.*, due to the continuous action of the applied force, will be

$$\frac{1}{m} \int_0^{t_1} F(t)(t_1 - t)dt, \tag{e}$$

and the complete displacement-time equation can be written in the form

$$x = x_0 + \dot{x}_0 t_1 + \frac{1}{m} \int_0^{t_1} F(t)(t_1 - t)dt. \tag{6}$$

The integral of expression (*e*) can be given a simple geometrical interpretation. Since $F(t)dt$ represents the area of one elemental strip in Fig.

4, we see that $F(t)dt(t_1 - t)$ is the *statical moment* of this area with respect to the ordinate at t_1 as an axis. Hence, the complete integral is simply the statical moment of the finite area of the force-time diagram between the ordinates $t = 0$ and $t = t_1$, said moment being taken with respect to the latter ordinate. Equation (6) is sometimes useful in studying the rectilinear motion of a particle under the action of a force $X = F(t)$.

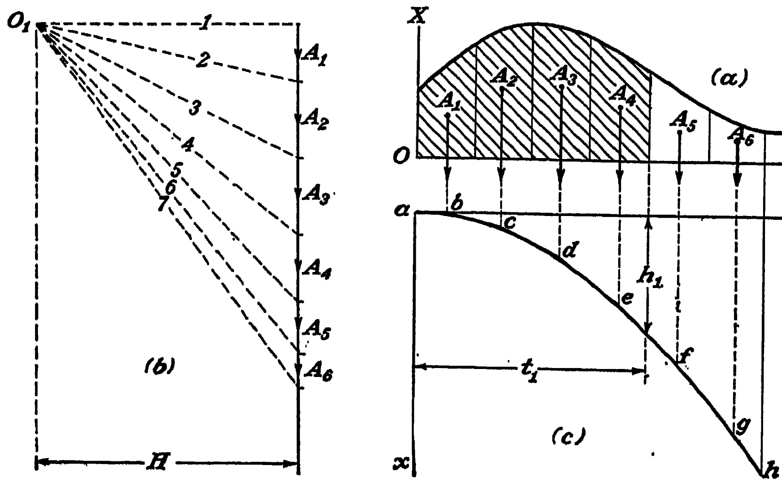


FIG. 6.

The above geometrical concept lends itself to a very simple graphical solution. Suppose, for example, that we want a displacement-time curve for the rectilinear motion of a particle of mass m under the action of a force $X = F(t)$ as represented by the force-time diagram in Fig. 6*a*. Then we pretend that the t-axis is a cantilever beam built-in on the right and carrying the force-time diagram as a distributed load. Replacing this distributed load by a series of concentrated forces proportional to the trapezoidal areas $A_1, A_2, A_3, \ldots$ and acting through their centroids, we construct a funicular polygon *abcd* . . . for this system of loads as shown in Fig. 6*c*. We know from graphic statics[1] that any intercept h_1 between this polygon and its first side *ab*, when multiplied by the pole distance H in Fig. 6*b*, represents the statical moment of the shaded area with respect to the ordinate at t_1 as an axis. Hence, this intercept, when multiplied by H and divided by m, represents the displacement at time t_1 due to the acting force $X = F(t)$. In short, the smooth curve inscribed in the polygon *abcde* . . . is the desired displacement-time curve.

If there is initial displacement x_0 and initial velocity $\dot{x}_0$, we have for the total displacement x at any time t_1

$$x = x_0 + \dot{x}_0 t_1 + \frac{Hh_1}{m}. \tag{f}$$

In evaluating displacements from this formula, we note from Fig. 6 that intercepts h_1 are measured in the same units as abscissa, *i.e.*, time. The pole distance H has the same dimension as fictitious force, *i.e.*, *force* $\times$ *time*.

[1] See the authors' "Theory of Structures," p. 30, McGraw-Hill, New York, 1945.

We come now to a consideration of the case where $X = F(x)$, as represented by the force-displacement diagram in Fig. 7. In this case, we can again obtain a solution of the differential equation of motion (1) by quadrature. First, we write

$$d\dot{x} = \frac{1}{m}F(x)dt.$$

Then multiplying both sides of this expression by $\dot{x} = dx/dt$, we obtain

$$\dot{x}\,d\dot{x} = \frac{1}{m}F(x)dx,$$

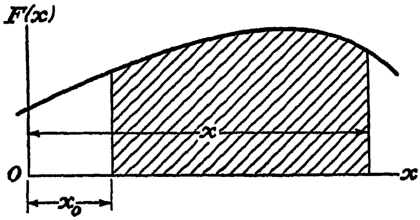


FIG. 7.

in which the new variables $\dot{x}$ and x are separated. Integrating, we get

$$\frac{\dot{x}^2}{2} + C = \frac{1}{m}\int_{x_0}^{x} F(x)dx.$$

Assuming now that we know the initial velocity $(\dot{x})_{x=x_0} = \dot{x}_0$, we find $C = -\dot{x}_0^2/2$ and the equation becomes

$$\frac{\dot{x}^2}{2} - \frac{\dot{x}_0^2}{2} = \frac{1}{m}\int_{x_0}^{x} F(x)dx. \tag{7}$$

This equation states that the change in kinetic energy of the particle over any portion of its path is equal to the corresponding work of the acting force as represented by the shaded area under the force-displacement diagram in Fig. 7. Equation (7) is very useful when we wish to express the velocity of a particle as a function of its displacement. Introducing the notation

$$G(x) = \int_{x_0}^{x} F(x)dx,$$

we have

$$\dot{x} = \pm\sqrt{\dot{x}_0^2 + \frac{2}{m}G(x)}, \tag{g}$$

where the sign is taken to agree with that of $\dot{x}_0$.*

Having velocity as a function of displacement from Eq. (g) we write

$$\frac{dx}{dt} = \dot{x},$$

from which

$$dt = \frac{dx}{\dot{x}}.$$

* If $\dot{x}_0 = 0$, we must choose the sign of dx/dt on the basis of the initial direction of the acting force $F(x)$ when $x = x_0$. If this force $F(x_0)$ is positive in direction, it will induce a positive increment of displacement dx in time dt and we have to take dx/dt with plus sign. If $F(x_0)$ is negative, dx/dt is taken with minus sign.

Then integrating, we obtain

$$t + C = \int_{x_0}^{x} \frac{dx}{\dot{x}}.$$

Since $t = 0$ when $x = x_0$, we have $C = 0$, so that finally

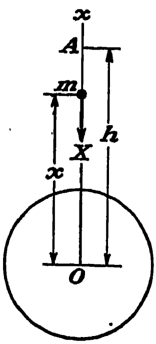


FIG. 8.

$$t = \int_{x_0}^{x} \frac{dx}{\dot{x}}. \tag{8}$$

From this expression, we obtain the displacement-time equation $x = f(t)$, and the motion is completely defined.

Example: A particle of mass m starts from rest at an initial height h above the center of the earth and falls under the action of gravity (Fig. 8). If air resistance is neglected and displacement x is measured from the center of the earth, the resultant acting force for any position of the particle above the earth will be[1]

$$X = -\frac{Wr^2}{x^2},$$

where W is the weight of the particle at the earth's surface and r is the radius of the earth. We are required to find the velocity of the particle as a function of its position x.

Solution: Using Eq. (7) and remembering that $\dot{x}_0 = 0$, we have

$$\frac{\dot{x}^2}{2} = -\frac{1}{m}\int_h^x \frac{Wr^2}{x^2}dx = g\left[\frac{r^2}{x}\right]_h^x = gr^2\left(\frac{1}{x} - \frac{1}{h}\right),$$

from which

$$\dot{x} = -\sqrt{2gr^2\left(\frac{1}{x} - \frac{1}{h}\right)}. \tag{h}$$

Taking $h = \infty$ and $x = r$, we obtain

$$\dot{x}_\infty = -\sqrt{2gr}. \tag{i}$$

This is the velocity with which a particle falling from infinity would strike the earth. With reversed sign, expression (i) also represents the initial velocity that must be given to a body at the earth's surface in order to have it completely escape the earth's gravitational field; it is called the *velocity of escape*. Taking $r = 3{,}960$ miles, we obtain $\dot{x}_\infty \approx 7$ m.p.s. This result, of course, is of no practical value, since our solution neglects the effect of air resistance.

To calculate the time of fall for the particle in Fig. 8, we substitute expression (h) for $\dot{x}$ in Eq. (8) and obtain

$$t = \frac{-h/r}{\sqrt{2gh}}\int_h^x \sqrt{\frac{x}{h-x}}\,dx. \tag{j}$$

To evaluate this integral, we let

$$x = h\cos^2\theta, \tag{k}$$

where $\theta = \cos^{-1}\sqrt{x/h}$ is a new variable. Then

$$dx = -2h\sin\theta\cos\theta\, d\theta$$

and Eq. (j) becomes

[1] This follows from Newton's law of gravitation, which states that the force of attraction between any two masses is inversely proportional to the square of the distance between their centers.

$$t = +\frac{2h^2/r}{\sqrt{2gh}}\int_0^\theta \cos^2\theta\, d\theta.$$

Integrating this, we get

$$t = \frac{h^2/r}{\sqrt{2gh}}\left(\theta + \frac{1}{2}\sin 2\theta\right). \tag{l}$$

We may now replace θ by $\cos^{-1}\sqrt{x/h}$ and obtain finally

$$t = \frac{h^2/r}{\sqrt{2gh}}\left[\cos^{-1}\sqrt{\frac{x}{h}} + \sqrt{\frac{x}{h}\left(1 - \frac{x}{h}\right)}\right]. \tag{m}$$

From this expression, the time of fall t can be calculated for any position x. The equation, of course, is valid only for $r < x < h$.

PROBLEMS

1. Under the influence of gravity and without resistance, particles start simultaneously from rest at O and slide along variously inclined straight grooves cut in the surface of a cone with inclined axis (Fig. 9). Prove that after any lapse of time t, all the particles will lie on the surface of a sphere of diameter $D = \frac{1}{2}gt^2$.

2. Assuming $\dot{x}_0 = x_0 = 0$, develop general velocity-time and displacement-time equations for the rectilinear motion of a particle of mass m under the action of a force $X = F_0e^{-\alpha t}$, where F_0 and α are constants.

3. Assuming $\dot{x}_0 = x_0 = 0$, develop general velocity-time and displacement-time equations for the rectilinear motion of a particle of mass m under the action of a force $X = F_0 \cos \omega t$, where F_0 and ω are constants.

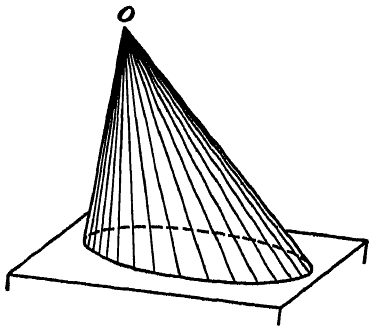


FIG. 9.

4. A car of mass m moving rectilinearly with initial velocity v_0 is brought to rest by a succession of negative impulses occurring at regular intervals of time $t = \tau, t = 2\tau, \ldots t = n\tau$. If each impulse reduces the velocity by the constant amount Δv, what number n of impulses will be required to bring the car to rest and what total distance x will it have traveled by this time? Assume, in addition to the impulses, a constant resistance R.

Ans. $n = \dfrac{v_0/\tau}{\Delta v/\tau + R/m}$; $x = \dfrac{v_0}{2}\left(\dfrac{v_0 + \Delta v}{\Delta v/\tau + R/m}\right)$.

5. Assuming as initial conditions $x = x_0$ and $\dot{x} = 0$ when $t = 0$, develop the displacement-time equation for the rectilinear motion of a particle of mass m under the action of a force $X = kx$.
Ans. $x = x_0\cosh \sqrt{k/m}\cdot t$.

2. Graphical Quadrature.—In the preceding article, we have seen that integration of the differential equation of motion (1) frequently reduces to two steps of simple quadrature. Thus if the acting force can be expressed analytically, there is no difficulty whatever. In engineering problems of dynamics, however, it frequently happens that the acting force is given graphically by an empirical curve like those shown in Figs. 4 and 7. In such cases, the quadratures indicated in the equations

of the previous article are best made graphically or numerically. We shall discuss here a graphical method of quadrature applicable to all such problems.

Referring to Fig. 10, assume that a function $\eta = f(\xi)$* is given by the curve AB and that we desire to determine graphically the integral function $\zeta = g(\xi) = \int_0^\xi f(\xi)d\xi$. This means simply that we have to construct a curve $\zeta = g(\xi)$ whose slope $d\zeta/d\xi$ for each value of ξ is equal

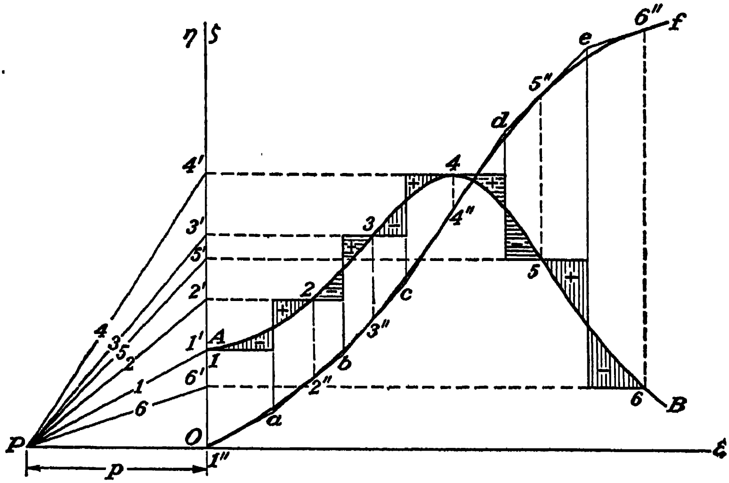


FIG. 10.

to the corresponding ordinate η of the given curve and whose successive ordinates ζ represent the accumulation of area under the given curve. To accomplish this construction, we proceed in the following manner: First, we select a suitable number of points 1, 2, 3, . . . 6 on the given curve AB and project these points onto the vertical η-axis, obtaining the corresponding points $1'$, $2'$, $3'$, . . . $6'$. Next, we select a pole P to the left of the origin O and any convenient distance p therefrom. Connecting the points $1'$, $2'$, $3'$, . . . $6'$ with the pole P, we obtain a series of rays 1, 2, 3, . . . 6, the slopes of which are all proportional to the corresponding ordinates of the given curve $\eta = f(\xi)$.

This done, we replace the given curve AB by a *step curve* passing horizontally through the chosen points 1, 2, 3, . . . 6 and so stepped that the areas of the similarly shaded triangles are as nearly equal as we can make them by eye. Considering this step curve for a moment, we see that since its ordinates are constant over each step, then the correspond-

* The discussion is perfectly general, and we use the notations ξ, η, ζ rather than x, y, z to avoid confusion with any specific quantities that x, y, z might suggest.

ing integral curve must have a constant slope over each step, as represented by the slopes of the rays 1, 2, 3, . . . 6. In other words, it must be a funicular polygon corresponding to these rays. To construct this polygon, we being at O and draw Oa parallel to ray 1; from a, where Oa intersects the first vertical of the step curve, we draw ab parallel to ray 2; etc., as shown in the figure. The smooth curve inscribed in the polygon $Oabcdef$ and tangent to it at the points 1″, 2″, 3″, . . . is the required integral curve.

This follows from two facts: (1) The step curve and the given curve AB have the points 1, 2, 3, . . . 6 in common; therefore, their integral curves must have the same slopes at these points. (2) Any ordinate of the integral curve at one of the points 1″, 2″, 3″, . . . 6″ represents the accumulation of area under the given curve AB up to that point. Likewise, it represents the accumulation of area under the step curve up to that point. For these areas to be one and the same, the areas of the similarly shaded triangles must be equal, a condition that was previously approximately fulfilled. Hence the smooth curve 1″, 2″, . . . 6″ and the true integral curve have at points 1″, 2″, . . . the same ordinates and the same tangents. With a proper selection of the points 1, 2, 3, . . . , our smooth curve will represent a very good approximation to the true integral curve.

One question remains to be discussed, namely: how to determine the scale by which the ordinates of the integral curve $\zeta = g(\xi)$ shall be measured. If the same scale unit were used for both the unit of η and the unit of ξ in constructing the given curve AB and, further, if this scale unit were taken as the pole distance p, then, of course, still the same scale unit would be used for measuring the ordinates ζ of the integral curve. However, it is usually neither convenient nor desirable to do this. In general, if e_ξ, e_η, and e_ζ units of length are the scale units[1] for ξ, η, and ζ, respectively, and the pole distance is p units of length, then the tangent of the angle that any ray in Fig. 10 makes with the ξ-axis, *i.e.*, the slope of any ray, is

$$\tan \phi_i = \frac{e_\eta \eta_i}{p}. \tag{a}$$

In the same way, the actual slope of any tangent to our integral curve is

$$\tan \phi_i = \frac{e_\zeta d\zeta}{e_\xi d\xi}. \tag{b}$$

[1] For example, if we are using a scale 1 in. = 7 f.p.s. on the ξ-axis, then $\frac{1}{7}$ in. = 1 f.p.s.; *i.e.*, $e_\xi = \frac{1}{7}$ in. In such case, ξ f.p.s. is represented by $e_\xi\xi$ in. of length in the drawing.

Equating expressions (a) and (b), we find

$$\frac{e_\eta}{p} = \frac{e_\zeta}{e_\xi}, \tag{c}$$

since, by definition and regardless of scale, we have $\eta = d\zeta/d\xi$. Thus it follows from expression (c) that the scale unit by which ordinates of the integral curve $\zeta = g(\xi)$ are to be measured is

$$e_\zeta = \frac{e_\xi e_\eta}{p}. \tag{9}$$

We shall now consider the application of the described method[1] to a specific problem of interior ballistics.[2] The curve OBC shown in Fig. 11 represents a typical *powder pressure curve* for a small-bore rifle; it shows the intensity of powder pressure as a function of the position of the bullet along the gun barrel. In this particular case, the bore is 8 mm., the bullet weighs 10 g., and the powder weighs 3.2 g. From these given data, it is required to find the so-called *muzzle velocity* v_c with which the bullet leaves the gun and also the time of travel t_c along the barrel. In making such calculations here, we shall assume that the resistance to motion along the barrel due to friction, rifling, etc., is negligible in comparison with the powder pressure. In other words, we assume that the force due to the expanding powder gas is the only acting force.

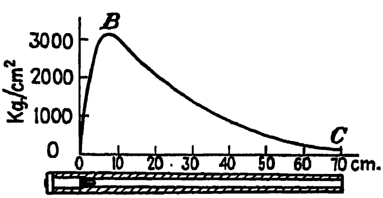


FIG. 11.

We begin our solution with Eq. (7) of the preceding article. Noting that in this case $\dot{x}_0 = x_0 = 0$, when $t = 0$, and using $v = \dot{x}$, we have

$$\tfrac{1}{2}m'v^2 = a\int_0^x p\,dx = ag(x), \tag{d}$$

wherein m' is usually taken to represent the mass of the bullet plus one-half of the powder mass, a is the cross-sectional area of the bore, and p is the powder pressure. From Eq. (d), we obtain

$$v = \sqrt{\frac{2a}{m'}}\sqrt{g(x)}, \tag{e}$$

and we see that by constructing graphically the integral curve $\zeta = g(x)$ corresponding to the given powder pressure curve $p = f(x)$, we can easily

[1] For a slight variation of the method as well as for other applications, see H. Von Sanden, "Practical Mathematical Analysis," Methuen, London, 1923.

[2] For a brief résumé of the principal problems of interior ballistics and bibliography, see "Innere Ballistik," by Erwin Bolle, "Handbuch der physikalischen und technischen Mechanik," vol. II, p. 276, Leipzig, 1930.

find therefrom the velocity v for any value of x. In short, the velocity is proportional to the square root of the work done by the powder pressure.

We proceed then by plotting the given powder pressure curve in Fig. 12a to the scale: 1 cm. = 1,000 kg. per sq. cm. on the p-axis and

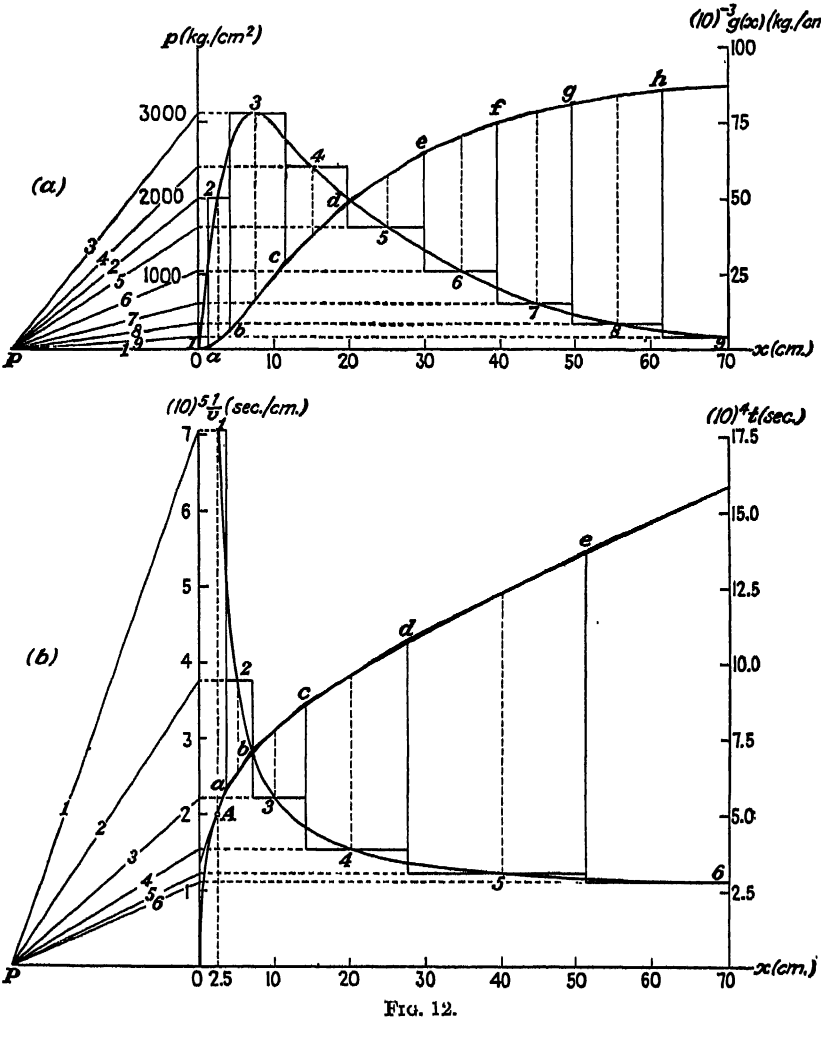


FIG. 12.

1 cm. = 10 cm. on the x-axis. Choosing the points 1, 2, 3, . . . 9 on this curve, we construct the corresponding step curve as already explained. Then taking a pole distance p = 2.5 cm., we construct the rays 1, 2, 3, . . . 9 and the corresponding funicular polygon $Oabcdefgh$ as shown. The smooth curve inscribed in this polygon gives us the

required integral curve $\zeta = g(x)$. Using Eq. (9), the scale factor for this curve is found to be

$$e_\zeta = \frac{1}{10} \times \frac{1}{1{,}000} \times \frac{1}{2.5} = \frac{1}{25{,}000},$$

i.e., on the ζ-axis, 1 cm. = 25,000 kg. per cm. The maximum ordinate at the end of this diagram scales 3.50 cm. Therefore $g_e(x) = 87{,}500$ kg. per cm. To find the corresponding muzzle velocity v_e, we have, for the given data,

$$\sqrt{\frac{2a}{m'}} = \sqrt{\frac{2 \times 0.5027 \times 980}{0.0116}} = 291.2 \text{ cm.}^{\frac{1}{2}} / \text{sec. kg.}^{\frac{1}{2}}$$

and then, from Eq. (*e*),

$$v_e = 291.2\sqrt{87{,}500} = 86{,}000 \text{ cm. per sec.} = 860 \text{ m. per sec.}$$

We now turn our attention to the time of travel of the bullet along the gun barrel. In the case of large guns, this question is of considerable interest in connection with the design of recoil mechanisms. Having velocity v as a function of displacement x, there is no very great difficulty to find the time of travel. We begin with Eq. (8) of the preceding article and write

$$t = \int_0^x \frac{dx}{v}. \qquad (f)$$

From this, we see that to find the time of travel to any point x along the barrel, we need only to construct the curve $1/v = \phi(x)$ and then make the corresponding integral curve $t = \psi(x)$ as represented by expression (*f*). To do this, we first scale a suitable number of ordinates of the $g(x)$-curve in Fig. 12*a* and calculate, from Eq. (*e*), the corresponding values of v and $1/v$ as shown in Table I. The next step is to plot the curve $1/v = \phi(x)$ as shown in Fig. 12*b*.

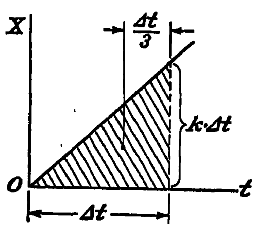


FIG. 13.

Because of the fact that this curve approaches infinity as x approaches zero, we cannot deal with it graphically for small values of x. In other words, the area under the early part of the curve is indeterminate. Therefore we solicit a little analytical aid at the beginning, namely: we assume that up to the point $x = 2.5$ cm., the powder pressure p is a linear function of time. Then the resultant force X acting on the bullet can be represented by the force-time diagram shown in Fig. 13. Using this diagram together with the notions of area and statical moment discussed on page 6, we may write at once

$$v_1 = \frac{k(\Delta t)^2}{2m'} \tag{g}$$

and

$$x_1 = \frac{k(\Delta t)^2}{2m'} \frac{\Delta t}{3}. \tag{h}$$

Eliminating the constant k/m', we obtain

$$\Delta t = 3\,\frac{x_1}{v_1}. \tag{i}$$

Using in this expression, the first pair of values of x and v from Table I, we find

$$\Delta t = \frac{3 \times 2.5}{14{,}860} = 5.05(10)^{-4} \text{ sec.} \tag{j}$$

To plot this starting point for the integral curve $t = \psi(x)$ in Fig. 12*b*, we must first determine the scale factor for t. Since we have plotted the

TABLE I

x cm.	$g(x)$ scale units	$g(x)$ kg./cm.	$\sqrt{g(x)}$ (kg./cm.)$^{\frac{1}{2}}$	$\sqrt{\frac{2a}{m'}}$	v cm./sec.	$(10)^5 \cdot \frac{1}{v}$ sec./cm.
0.0	0.000	0	0	291.2	0	∞
2.5	0.104	2,600	51.0	↓	14,860	6.750
5.0	0.330	8,325	91.1		26,500	3.774
7.5	0.652	16,300	127.7		37,100	2.695
10	0.945	23,600	153.6		44,700	2.237
20	1.934	48,400	220.0		64,000	1.562
30	2.586	64,600	254.2		74,000	1.351
40	2.996	74,900	273.7		79,600	1.256
50	3.255	81,300	285.1		83,000	1.205
60	3.434	85,800	292.9		85,300	1.172
70	3.500	87,500	295.8		86,000	1.163

curve $1/v = \phi(x)$ to the scale 1 cm. = 0.00001 sec. per cm. on the vertical axis and 1 cm. = 10 cm. on the x-axis, we have $e_{1/v} = 100{,}000$ and $e_x = \frac{1}{10}$, so that with a pole distance $p = 2.5$ cm., we have

$$e_t = 100{,}000 \times \frac{1}{10} \times \frac{1}{2.5} = 4{,}000;$$

i.e., on the t-axis, 1 cm. = $2.5(10)^{-4}$ sec. We now plot the point $A[t = 5.05(10)^{-4}$ sec. when $x = 2.5$ cm.] in Fig. 12*b* and are ready to begin the graphical construction of the integral curve $t = \psi(x)$. This is carried out in the same manner as before, and the final ordinate at

$x = 70$ cm. scales 6.35 cm. Hence,

$$t_c = 6.35 \times 2.5(10)^{-4} = 1.59(10)^{-3} \text{ sec.}$$

PROBLEMS

6. For the particular case where $h = 2r = 7{,}920$ miles, construct by graphical quadrature the integral function $t = g(x)$ as represented by Eq. (j) in Art. 1 and check the accuracy of the graphical method by using Eq. (m).

7. A test made on a hunter's longbow (Fig. 14) shows the following relation between the draw d in inches and the pull P in pounds:

Draw d	6	10	15	20	25	30
Pull P	0	8.6	19.5	32.1	47.0	70.0

Plot the corresponding curve $P = F(x)$, and determine, by graphical integration, the maximum velocity with which a $\frac{3}{4}$-oz. arrow leaves the bow for full draw $d = 30$ in.

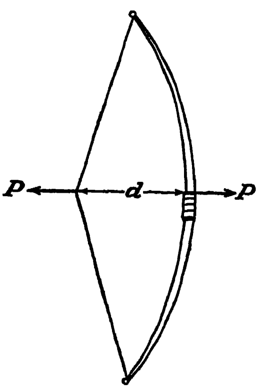


FIG. 14.

3. Numerical Quadrature.—In dealing with the integral

$$g(\xi) = \int_0^\xi f(\xi)d\xi,$$

it often happens that the given function $\eta = f(\xi)$ is defined simply by a series of numerical values of η over a given range of ξ. These values of the function may have been observed experimentally, scaled from a graph, or computed from a given analytic function. In any case, it is frequently desirable to work directly with the given numerical data rather than to plot the function to scale and then work graphically or to attempt an exact integration. In this article, we shall describe and illustrate a common method of *numerical quadrature*.

Referring to Fig. 15, let the function $\eta = f(\xi)$ be defined within the region $(\xi - \xi_0)$ by the ordinates η_0, η_1, η_2, . . . η_n, for a series of equidistant values of ξ: ξ_0, $\xi_0 + \Delta\xi$, $\xi_0 + 2\Delta\xi$, . . . $\xi_0 + n\Delta\xi$. Then the problem of evaluating the integral function becomes simply one of finding the area under the curve $\eta = f(\xi)$ between the initial ordinate η_0 and successive ordinates η_1, η_2, . . . η_n. The simplest approximation to this area would be obtained by assuming that the function is linear between the given points and taking successive

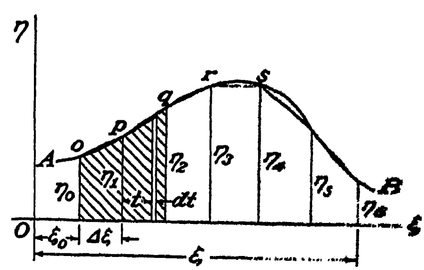


FIG. 15.

sums of the areas of the trapezoids shown in the figure. Thus

$$g(\xi) \approx \frac{\Delta\xi}{2}(\eta_0 + \eta_1) + \frac{\Delta\xi}{2}(\eta_1 + \eta_2) + \frac{\Delta\xi}{2}(\eta_2 + \eta_3) + \cdots. \tag{10a}$$

If we do not care about intermediate values of the integral function but require only the total area under a definite portion AB of the curve, we have

$$\int_A^B f(\xi)d\xi \approx \Delta\xi(\tfrac{1}{2}\eta_0 + \eta_1 + \eta_2 + \cdots + \eta_{n-1} + \tfrac{1}{2}\eta_n). \tag{10b}$$

Formula (10) is called the *trapezoid rule*. It is often very helpful in making a quick rough approximation to the area under a given curve.

A much better approximation to the area under the curve AB in Fig. 15 will be obtained by assuming that over any double interval $2\Delta\xi$, such a portion of the curve as opq is a parabola given by the equation

$$\eta = A + Bt + Ct^2, \tag{a}$$

where A, B, and C are constants and t is measured from the middle ordinate η_1 as shown in the figure. The constants A, B, C can always be selected so as to make the parabola pass through the three given points o, p, q. However, instead of doing this, we find the values of the ordinates η_0, η_1, η_2 in terms of the constants A, B, C by substituting $t = -\Delta\xi$, 0, $+\Delta\xi$ into Eq. (a). In this way, we find that

$$\eta_0 + 4\eta_1 + \eta_2 = 6A + 2C(\Delta\xi)^2. \tag{b}$$

At the same time, the area under the portion opq of the curve (shaded in the figure) is

$$S_1 = \int_{-\Delta\xi}^{+\Delta\xi} \eta\, dt.$$

Substituting expression (a) for η and integrating, we find

$$S_1 = 2A\,\Delta\xi + 2C\frac{(\Delta\xi)^2}{3} = \frac{\Delta\xi}{3}[6A + 2C(\Delta\xi)^2]. \tag{c}$$

Then from Eqs. (b) and (c) together, we obtain

$$S_1 = \frac{\Delta\xi}{3}(\eta_0 + 4\eta_1 + \eta_2). \tag{11}$$

Using expression (11), successively, for the portions opq, qrs, . . . of the given curve AB, we find for the total area up to any even-numbered ordinate

$$g(\xi) = \frac{\Delta\xi}{3}(\eta_0 + 4\eta_1 + \eta_2) + \frac{\Delta\xi}{3}(\eta_2 + 4\eta_3 + \eta_4) + \frac{\Delta\xi}{3}(\eta_4 + 4\eta_5 + \eta_6) + \cdots. \tag{12a}$$

Again, if we do not care about intermediate values of the integral function but want only the total area under a specified portion AB of the given curve, we have

$$\int_A^B f(\xi)d\xi = \frac{\Delta\xi}{3}(\eta_0 + 4\eta_1 + 2\eta_2 + 4\eta_3 + 2\eta_4 + \cdots + \eta_n). \quad (12b)$$

Formula (12) is called *Simpson's rule.* In general, it gives a much greater accuracy than the trapezoid rule, and its use involves very little additional effort. It must be noted that in using formula (12b), the given region AB must be divided into an *even* number of *equal* intervals $\Delta\xi$. In other words, the given function must be defined by an *odd* number of ordinates equally spaced along the ξ-axis.

Still greater accuracy can be obtained by approximating a given curve by a polynomial of the nth degree, but the formulas for calculating the area under such a curve are considerably more cumbersome than Simpson's rule, and the slight increase in accuracy is often not worth the additional labor.[1] Furthermore, in working with empirical data, such refinement is usually unjustified.

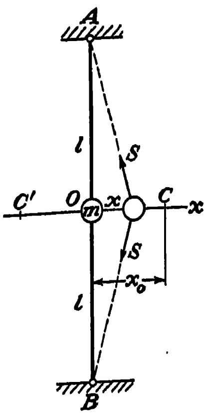


FIG. 16.

We shall now discuss the application of our numerical integration to a problem in the field of vibrations. In Fig. 16, a weight $W = mg$ is attached at the mid-point of a tightly stretched piano wire AB. If this weight be displaced laterally from its position of equilibrium and released, it will, in the absence of damping, vibrate with a constant amplitude equal to the initial displacement x_0. Of particular interest will be the period of such vibration, *i.e.*, the time required to complete one cycle of the motion from C to C' and back to C again. Since the system is symmetrical with respect to the middle configuration, this period will obviously be four times the time required to pass from C to O. Hence, in our further discussion we confine our attention to the first quarter cycle of vibration. We begin with the following list of notations:

$2l$ = total length of wire,
S_0 = initial tensile force in wire,
x_0 = initial lateral displacement,
A = cross-sectional area of wire,
E = modulus of elasticity of wire in tension,
m = mass of attached ball.

[1] For a discussion of more elaborate methods of numerical quadrature, see H. Von Sanden, "Practical Mathematical Analysis," p. 18.

From simple geometrical considerations, we see that the tensile strain in the wire due to displacement x is

$$\frac{\sqrt{l^2 + x^2} - l}{l} \approx \frac{x^2}{2l^2}. \tag{d}$$

Then the total tension in the wire is

$$S \approx S_0 + AE\frac{x^2}{2l^2}. \tag{e}$$

Since there are two of these forces (see Fig. 16), the resultant force on the displaced particle is directed along the x-axis and has the magnitude

$$X = -2S\left(\frac{x}{\sqrt{l^2 + x^2}}\right) \approx -\frac{2S_0}{l}x - \frac{AE}{l^3}x^3 = F(x). \tag{f}$$

From this expression, we see that if the initial tension S_0 is very large and the displacement x is always small, then the first term predominates and we have a restoring force that is approximately a linear function of displacement. In such case, the vibrations will be simple harmonic and the period τ is independent of amplitude. Otherwise, the restoring force is definitely not a linear function of displacement but increases in greater proportion and the period of vibration will be somewhat diminished for large amplitudes. In any particular case, the period of vibration for a given amplitude can be calculated from Eqs. (7) and (8) of Art. 1.

We begin with Eq. (7), which for $\dot{x}_0 = 0$ becomes

$$\dot{x}^2 = \frac{2}{m}\int_{x_0}^{x} F(x)dx. \tag{g}$$

To make the indicated quadrature numerically, we take definite numerical data as shown in Table II and evaluate the function $F(x)$ as given by Eq. (f) for a series of chosen values of x. The function $-F(x)$ plotted from this table is shown graphically in Fig. 17.

We are now ready to proceed with our numerical integration. Using the values of $-F(x)$ as given in the second column of Table III, together with formula (11), we make successive steps of quadrature, evaluating each time the integral $\int_{x_i}^{x_i+2\Delta x} F(x)dx$ as shown in column (4).[1] In column (5), these increments of area are multiplied by the factor $2/m = 100$, in accordance with Eq. (g), and successive summations of these incre-

[1] The first step from $x = 1.25$ to $x = 1.20$ is made by the trapezoid rule. Since both $F(x)$ and dx in Eq. (g) are negative, the integral is positive and we ignore signs in the table.

ments of $\dot{x}^2$ are given in column (6). Finally, in column (7), we have the final results of the first integration, namely: $-\dot{x} = \phi(x)$.* This function is also shown graphically in Fig. 17.

TABLE II

x in.	$\frac{2S_0}{l}x$ lb.	x^3 in.[3]	$\frac{AE}{l^3}x^3$ lb.	$-F(x)$ lb.	$-F(x) = \frac{2S_0}{l}x + \frac{AE}{l^3}x^3$
1.25	25.0	1.953	31.85	56.85	$x_0 = 1.25$ in.
1.20	24.0	1.728	28.20	52.20	
1.10	22.0	1.331	21.710	43.71	$S_0 = 200$ lb.
1.00	20.0	1.000	16.310	36.31	
0.9	18.0	0.729	11.890	29.89	$l = 20.0$ in.
0.8	16.0	0.512	8.353	24.35	
0.7	14.0	0.343	5.595	19.60	$2S_0/l = 20.0$ lb. per in.
0.6	12.0	0.216	3.523	15.52	
0.5	10.0	0.125	2.038	12.04	$A = 0.00435$ sq. in.
0.4	8.0	0.064	1.044	9.04	
0.3	6.0	0.027	0.441	6.44	$E = 30(10)^6$ p.s.i.
0.2	4.0	0.008	0.130	4.13	
0.1	2.0	0.001	0.016	2.02	$AE/l^3 = 16.3145$ lb. per cu. in.
0.0	0.0	0.000	0.000	0.00	$m = 0.0200$ lb.-sec.^{2} per in.

Having $-\dot{x} = \phi(x)$ as represented by the data in columns (1) and (7) of Table III, we are now ready to consider the time integral[1]

$$t = \int_{x_0}^{0} \frac{dx}{\dot{x}}. \tag{h}$$

As a preliminary to the evaluation of this integral, values of $-1/\dot{x}$ are recorded in column (8) of Table III. Here again, the area under the initial portion of the $-1/\dot{x}$ curve makes difficulty, since for $x_0 = 1.25$ in., $-1/\dot{x}_0 = \infty$. To get around this difficulty, we assume that from $x_0 = 1.25$ in. to $x_1 = 1.20$ in., the particle moves with constant acceleration a. Then

$$\Delta x_1 = \frac{1}{2}a(\Delta t)^2 = \frac{v_1 \Delta t}{2},$$

from which

$$\Delta t = \frac{2\Delta x_1}{v_1} = \frac{2(0.05)}{16.51} = 0.006058 \text{ sec.} \tag{i}$$

Now beginning our numerical integration at $x_1 = 1.20$ in. and pro-

* The intermediate values shown in boldface type are found by interpolation.

[1] Again, both $\dot{x}$ and dx are negative; t is positive, and we ignore signs in the table.

TABLE III

(1) x in.	(2) $-F(x)$ lb.	(3) $(\eta_0 + 4\eta_1 + \eta_2)$ lb.	(4) $\frac{\Delta x}{3} \cdot (\cdot\cdot\cdot)$ lb.-in.	(5) $\frac{2\Delta x}{3m}(\cdot\cdot\cdot)$ (in./sec.)^2	(6) $\dot{x}^2$ (in./sec.)^2	(7) $-\dot{x}$ in./sec.	(8) $-\frac{1}{\dot{x}}$ sec./in.	(9) $(\eta_0 + 4\eta_1 + \eta_2)$ sec./in.	(10) $\frac{\Delta x}{3} \cdot (\cdot\cdot\cdot)$ sec.	(11) t sec.
1.25	56.85	......	.....	.....	....	0	∞			
1.20	52.20	....	2.726	272.6	272.6	16.51	0.06060	$\Delta t = \frac{2x_1}{v_1} = \frac{2(0.05)}{16.51} = 0.006058$		
1.10	43.71	......	.....	.....	....	**27.50**	0.03636			
1.00	36.31	263.35	8.778	877.8	1,150.4	33.92	0.02949	0.23553	0.007851	0.01391
0.9	29.89	......	.....	.....	....	**38.69**	0.02654			
0.8	24.35	180.22	6.007	600.7	1,751.1	41.85	0.02390	0.15955	0.005318	0.01923
0.7	19.60	......	.....	.....	....	**44.43**	0.02251			
0.6	15.52	118.27	3.942	394.2	2,145.3	46.33	0.02158	0.13552	0.004517	0.02374
0.5	12.04	......	.....	.....	....	**47.80**	0.02093			
0.4	9.04	72.72	2.424	242.4	2,387.7	48.87	0.02047	0.12577	0.004192	0.02794
0.3	6.44	......	.....	.....	....	**49.66**	0.02014			
0.2	4.13	38.93	1.298	129.8	2,517.5	50.20	0.01992	0.12095	0.004032	0.03197
0.1	2.02	......	.....	.....	....	**50.50**	0.01980			
0.0	0.00	12.21	0.407	40.7	2,558.2	50.60	0.01977	0.11889	0.003963	0.03593

ceeding as before, we find for the even values of x in column (1) the corresponding values of t as given in column (11). The displacement-time curve $t = \psi(x)$ plotted from these data is shown in Fig. 17.

As already noted, the system is symmetrical with respect to the middle configuration, and we have therefore, for the period of vibration,

$$\tau = 4(0.03593) = 0.1437 \text{ sec.} \tag{j}$$

From Fig. 17, we see that for small amplitudes ($x_0 < 0.2$ in.), we can assume a linear restoring force and take

$$X = -\frac{2S_0}{l}x. \tag{f'}$$

In such case, the period is

$$\tau = 2\pi\sqrt{\frac{lm}{2S_0}} = 0.1985 \text{ sec.} \tag{k}$$

Comparing expressions (j) and (k), we see that for the assumed larger amplitude, the period of vibration is reduced by about 28 per cent.

In columns (9), (10), and (11) of Table III, we have proceeded in such a way as to obtain sufficient data for plotting the displacement-time

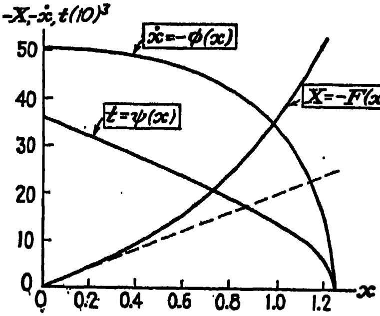


FIG. 17.

curve $t = \psi(x)$ shown in Fig. 17. If we are interested only in the period of vibration for a given amplitude, we can save some time by using formula (12b) to evaluate the time integral (h). In such case, using the data in column (8), we have

$$\tau_{1.25} = 4\left[\frac{2(0.05)}{16.5} + (0.0606 + 0.1454 + 0.0590 + 0.1062 + 0.0478 + 0.0900 + 0.0432 + 0.0837 + 0.0409 + 0.0806 + 0.0398 + 0.0792 + 0.0198) \times 0.0333\right] = 0.1437 \text{ sec.}$$

Making similar calculations for several different amplitudes, it is pos-

sible to construct a curve showing how the period of vibration varies with the amplitude. For the above system, such a curve is shown in Fig. 18.

PROBLEMS

8. Show that for $X = F(x)$ as defined by Eq. (*f*), the exact integration of Eq. (*g*) gives $-\dot{x} = \sqrt{K_1(x_0^2 - x^2) + K_2(x_0^4 - x^4)}$, where $K_1 = 2S_0/lm$ and $K_2 = AE/2l^3m$. Compare the values of $\dot{x}$ computed from this exact formula with those obtained by numerical quadrature in column (7) of Table III.

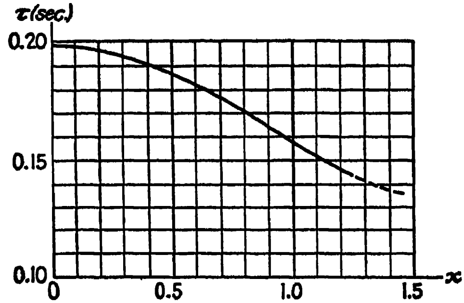


FIG. 18.

9. A .30-caliber rifle (bore diameter = 0.30 in.) has a 30-in. barrel length, a maximum powder pressure p_{max} = 45,600 p.s.i. that occurs when $x = x_1$ = 4.17 in., and shoots a 150-grain bullet with 50 grains of powder. The complete powder pressure curve is defined by the numerical data shown in the accompanying table. Calculate the muzzle velocity v and the time of travel t by numerical quadrature.

$\frac{x}{x_1}$	$\frac{p}{p_{max}}$	$\frac{x}{x_1}$	$\frac{p}{p_{max}}$	$\frac{x}{x_1}$	$\frac{p}{p_{max}}$
0	0	2.2	0.612	4.8	0.228
0.1	0.2	2.4	0.559	5.0	0.215
0.2	0.41	2.6	0.512	5.2	0.202
0.3	0.61	2.8	0.470	5.4	0.190
0.4	0.76	3.0	0.432	5.6	0.178
0.6	0.92	3.2	0.398	5.8	0.167
0.8	0.98	3.4	0.369	6.0	0.156
1.0	1.00	3.6	0.343	6.2	0.145
1.2	0.946	3.8	0.319	6.4	0.134
1.4	0.870	4.0	0.297	6.6	0.124
1.6	0.798	4.2	0.277	6.8	0.114
1.8	0.731	4.4	0.259	7.0	0.105
2.0	0.669	4.6	0.243	7.2	0.096

These data were taken from C. Cranz, "Lehrbuch der Ballistik," Vol. II, p. 329, Springer, Berlin, 1926.

10. What effect will a change in the weight of the bullet from 150 to 180 grains have on the muzzel velocity v and time of travel t in the preceding problem if all other data remain unchanged?

4. Rectilinear Motion in a Resisting Medium.—Whenever a body moves through a fluid medium such as air or water or along a rough track, it encounters a certain amount of resistance which, in the absence of any other forces, would eventually bring it to rest. Thus, a pendulum swinging freely in air gradually loses energy and finally comes to rest in its position of equilibrium. Likewise, a ship or train that cuts off its power at full speed will glide gradually to a stop owing to various resisting forces. Sometimes such resisting forces are so small compared with other forces that may act upon a body that they can be neglected without serious error. In other cases, they are of primary importance and have a pronounced effect upon the motion. As examples of problems in which resistance plays an important role, we may mention the motion of ships through water and the motion of aircraft and projectiles through the air. The slow settlement of fine particles of silt in a quiet body of water or of particles of volcanic dust in the atmosphere is an example of rectilinear motion in which the resisting forces predominate.

The exact nature of the resistance encountered by a body moving through a given fluid medium is rather complex; it usually involves several completely different physical phenomena. In general, resistance may be divided into four types: (1) *frictional resistance*, due to ordinary Coulomb friction on track or guides; (2) *viscous resistance*, due to the viscosity of the fluid medium in which a body moves; (3) *wake resistance*, due to eddy formations left in the fluid medium behind the moving body; and (4) *wave resistance*, due to the formation of waves radiating from the body into the surrounding medium. When a body moves through a stationary medium or, what amounts to the same thing, when a fluid medium flows past a stationary obstruction, the streamlines are dammed up in front of the body and exert a dynamical pressure on its nose. In going around the body, the streamlines are separated and do not always close up behind, so that a wake of eddy formations is left downstream. As a result of this, the pressures on the rear half of the body are appreciably less than on the front half; consequently, there is a resultant force on the body opposing its motion relative to the stream. This is the resistance referred to as wake resistance. In the case of ships moving on the surface of a liquid, there is another type of pressure resistance due to the formation of surface waves. Such wave resistance is also important in the case of projectiles moving through the air at speeds greater than that of sound.

While the various sources of resistance mentioned above are essentially different in character and extremely complex in nature, they may be said to have one thing in common, namely: they simply represent different modes of transference of energy from the moving body to the sur-

rounding medium where it is eventually reduced to heat. For this reason, the forces of resistance are often referred to as *dissipative forces.*

We come now to the all-important question of how to express the total resisting force. That it is some function of the velocity of the body relative to the fluid medium and that it always opposes the motion must be self evident. Thus, in general, we may write

$$R = -Ckf(v), \tag{13}$$

where k is a physical constant, the value of which depends upon the properties of the fluid medium and the size of the body, and C is a numerical factor, called the *coefficient of resistance.* The value of C depends principally upon the size and shape of the body and sometimes also upon its velocity.

In many cases, where *frictional resistance* predominates, a satisfactory form for the resistance function (13) will be

$$R = \text{const.} \tag{13a}$$

This law applies in the case of a body sliding over a clean dry surface, in which case $R = -fN$; f being the coefficient of friction and N the normal force.

If sliding surfaces are separated by a thin film of lubricant and the velocities are not too large, *viscous resistance* will predominate. In this case, both theory and experiment verify the fact that such resistance is proportional to the first power of the velocity. Thus

$$R = -Kv. \tag{13b}$$

This law applies also to the motion of very small spherical particles through a resisting medium such as water or air. In such case, if d is the diameter of the sphere and μ the absolute or *dynamic viscosity* of the fluid medium, Eq. (13b) takes the form

$$R = -3\pi\mu vd \tag{13b'}$$

which is known as *Stokes' law.* This law holds only for values of Reynolds number $N_R = vd/\nu < 1.0$, where, in general, d = a characteristic lateral dimension of the body and $\nu = \mu/\rho$ = *kinematic viscosity* of the fluid; ρ being the mass density of the fluid and μ the dynamic viscosity.

For other than minute velocities through a fluid medium and in the absence of any kind of sliding resistance, the major source of resistance lies in the unbalanced pressure distribution over the surface of the moving body. For such cases, Newton was the first to propose a definite form for the resistance function (13). He reasoned that in unit time, a body of cross-sectional area a, normal to the line of motion and moving with

velocity v, would have to impart its own speed to a mass of fluid, of mass density ρ, equal to ρav. Then, since the force necessary to do this is proportional to the rate of change of momentum, *i.e.*, to $(\rho av)v$, he concluded that resistance to motion of a given body through a given fluid medium is proportional to the square of the velocity. It is customary now to write this so-called *quadratic law of resistance* in the form

$$R = -C\tfrac{1}{2}\rho av^2 = -Ckv^2. \tag{13c}$$

While the whole mechanism of such resistance is much more complex than Newton supposed, modern hydrodynamical research has shown the quadratic law of resistance to be the best form for a variety of cases, provided the resistance coefficient C is properly evaluated and provided we are willing to accept the fact that it is not always a constant as New-

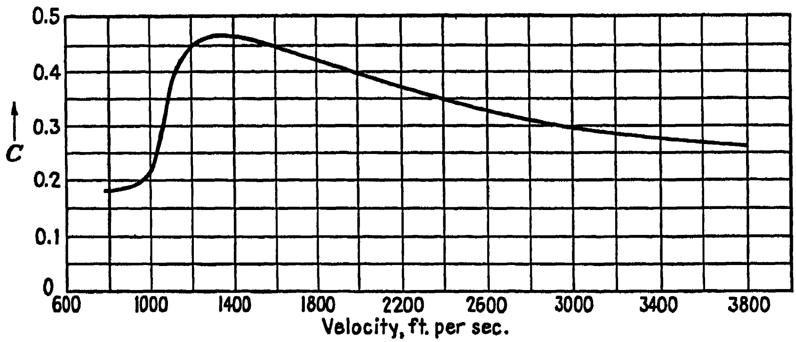


FIG. 19.

ton supposed. A detailed discussion of the coefficient C properly belongs to the field of fluid mechanics,[1] and we can give here only a few brief remarks about it.

In those cases where wake resistance predominates, the coefficient of resistance C in Eq. (13c) may be taken as a constant. For motion in air below the velocity of sound, this will apply to short, blunt, sharp-edged bodies such, for example, as a circular disk flatwise to the air stream. In this case, $C = 1.12$. It also applies, with good accuracy, to very rough elongated bodies such as a locomotive or blunt-nosed projectile. Up to moderate speeds, it may apply even to the motion of marine ships, although at higher speeds, where wave resistance predominates, it no longer does so.

In the case of a smooth, streamlined body such as an airship, both viscous resistance and wake resistance are important and the coefficient C varies with the velocity or, more properly, it is a function of Reynold's

[1] For an excellent discussion of the whole question of resistance, see Prandtl, "The Physics of Solids and Fluids," 2d ed., pt. II, pp. 285–299, 378, Blackie, Glasgow, 1936.

number. These cases are the most complex and will not be discussed here.[1]

In the case of a projectile moving with velocities above the speed of sound, denoted by c, wave resistance predominates and the coefficient C in Eq. (13c) can be represented as a function of velocity. A typical resistance curve[2] for a rifle bullet is shown in Fig. 19, where the coefficient of resistance C in Eq. (13c) is plotted against the velocity v. We note that as we pass from subsonic to supersonic velocities, there is a marked rise in the resistance coefficient, which for very high velocities tends again toward a constant value.

In the case of marine ships, all three types of fluid resistance are important and it is common practice simply to represent the total resistance as a function of velocity by an experimental curve.[3] Such a resist-

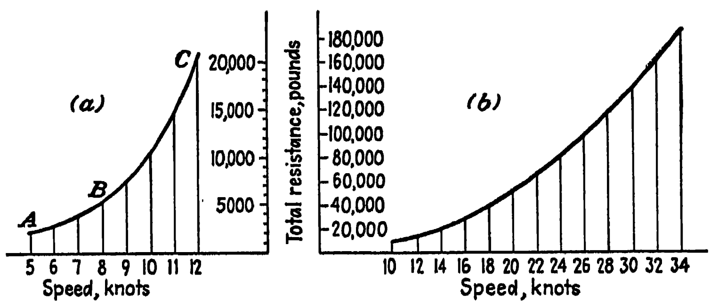


FIG. 20.

ance curve for a full-sized ship[4] is shown in Fig. 20a, no data being given for speeds below 5 knots. The portion AB of this curve shows the resistance to be proportional to the square of the speed, while in the neighborhood of point C it is more nearly proportional to the fourth power of the speed. The curve in Fig. 20b represents the resistance for a modern high-speed ship, such as a destroyer.

We come now to a consideration of the differential equation of rectilinear motion under the action of a force involving the resistance

[1] *Ibid.*, p. 299.

[2] This curve has been taken from an article on Exterior Ballistics by Otto von Eberhardt. See "Handbuch der physikalischen und technischen Mechanik," vol. II, p. 199.

[3] The experimental data are usually obtained for a small-scale model and then adapted to the prototype in accordance with Froude's law of similarity.

[4] The curve in Fig. 20a is of considerable historical interest. It was made in 1871 for H.M.S. "Greyhound" (length, 172 ft. 6 in., breadth, 33 ft. 2 in., draught, 13 ft. 9 in., displacement, 1,161 tons) by William Froude and represents the first serious attempt to study the nature of resistance of surface ships. See *Trans., Inst. Naval Architects*, 1874; also, *Naval Science*, vol. III; and *Engineering*, May 1, 1874.

function (13). To make the discussion more general, we assume that there is also a constant force and take

$$X = X_0 - Ckf(v). \tag{a}$$

Then the equation of motion (1) becomes

$$m\frac{dv}{dt} = X_0 - Ckf(v),$$

which we may write in the form

$$\frac{dv}{dt} = a - \beta f(v), \tag{b}$$

where $a = X_0/m$ is the acceleration produced by the constant force X_0 and $\beta = Ck/m$ is a resistance factor that is constant if C is constant. In any case, β may be regarded as a function of velocity, so that the entire right-hand side of Eq. (*b*) is a function of velocity and the variables are separable. By quadrature then, we obtain

$$t = \int_{v_0}^{v} \frac{dv}{a - \beta f(v)} \tag{14}$$

and, since $dx = v\,dt$, also

$$x = \int_{v_0}^{v} \frac{v\,dv}{a - \beta f(v)}. \tag{15}$$

After evaluation of these two integrals, for any given case, the variable v may be eliminated (or used as a parameter) and we have $x = f(t)$, which completely defines the motion.

Referring back to Eq. (*b*), we see that when the velocity has such a value V that

$$f(V) = \frac{a}{\beta}, \tag{c}$$

the acceleration vanishes and there is no further increase in velocity. The velocity V that satisfies Eq. (*c*) is called the *limiting velocity* or *terminal velocity*. Thus, for example, a ship or train under full power accelerates until this terminal velocity is reached and thereafter runs at this constant speed, the power developed by the engine being completely used up in overcoming the resistance.

We may now discuss the solution of Eqs. (14) and (15) for several particular cases. We begin with the case where viscous resistance predominates and the resistance is proportional to the first power of the velocity [Eq. (13*b*)]. In such case, the equations become

$$t = \int_{v_0}^{v} \frac{dv}{a - \beta v} \quad (14a) \qquad \text{and} \qquad x = \int_{v_0}^{v} \frac{v\,dv}{a - \beta v}, \quad (15a)$$

where $a = X_0/m$ and $\beta = K/m$ are constants. Evaluating the first of these integrals, we obtain

$$t = \frac{1}{\beta} \log \frac{a - \beta v_0}{a - \beta v}, \tag{d}$$

which may be written in the equivalent form

$$v = \frac{a}{\beta} - \left(\frac{a}{\beta} - v_0\right) e^{-\beta t}. \tag{e}$$

This is the general velocity-time equation for the motion. Replacing v by dx/dt and integrating again, we obtain, for $x_0 = 0$,

$$x = \frac{at}{\beta} - \frac{1}{\beta}\left(\frac{a}{\beta} - v_0\right)(1 - e^{-\beta t}), \tag{f}$$

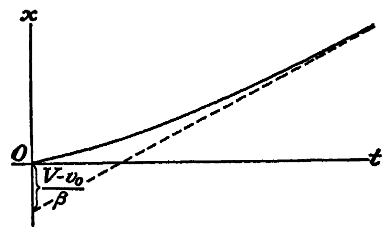


FIG. 21.

representing the final displacement-time equation. It will be noted from Eq. (*e*) that the velocity v approaches the terminal velocity $V = a/\beta$ as a limit but theoretically requires infinite time to reach this speed. The general nature of the displacement-time curve, plotted from Eq. (*f*), is shown in Fig. 21. We see that for large values of t, we may replace Eq. (*f*) by the equation of its asymptote

$$x = Vt - \frac{V - v_0}{\beta}. \tag{g}$$

This amounts to assuming a uniform motion with a fictitious initial displacement as shown by the dotted line in Fig. 21.

In the particular case where the viscous resistance $-Kv$ is the only acting force, we put $a = 0$ in Eqs. (*e*) and (*f*) and obtain

$$\left.\begin{aligned} v &= v_0 e^{-\beta t}, \\ x &= \frac{v_0}{\beta}(1 - e^{-\beta t}). \end{aligned}\right\} \tag{h}$$

From these equations, we see that for this kind of resistance, theoretically infinite time is required to bring the body to rest but the distance covered is finite, being v_0/β.

Example: Equations (*e*), (*f*), and (*g*) apply to the settlement of fine particles of silt in water or mist in the atmosphere. In such cases, we use Stokes's law of resistance for smooth spheres

$$R = -3\pi\mu\, vd = -Kv, \tag{i}$$

where μ is the dynamic viscosity of the fluid medium and d is the diameter of the particle (assumed spherical). If w_1 is the specific weight of the particle and w_2 the

specific weight of the fluid, the apparent gravity $a = g(w_1 - w_2)/w_1$ and

$$\beta = \frac{K}{m} = \frac{18\mu g}{w_1 d^2}.$$

Thus the terminal velocity becomes

$$V = \frac{a}{\beta} = \frac{g(w_1 - w_2)w_1 d^2}{w_1 18\mu g} = \frac{w_1 - w_2}{18\mu} d^2. \tag{j}$$

For particles of silt (specific gravity = 3) in water at 72°F. ($\mu = 2 \times 10^{-5}$ lb.-sec. per sq. ft.),

$$V = \frac{2(62.4)}{18 \times 2 \times 10^{-5}} d^2 = 3.46 \times (10)^5 d^2.$$

Taking $d = 1/7{,}000$ ft. (this corresponds to Reynolds number $N_R = 0.3$), the terminal velocity is $V = 0.00706$ f.p.s. Assuming $v_0 = 0$ and using Eq. (g), the time required for such a particle to settle to the bottom of a reservoir 100 ft. deep would be 14,200 sec., or approximately 4 hr.

In using the quadratic law of resistance in combination with a constant force, it is necessary to distinguish between two cases: (1) The constant force is a driving force ($a > 0$), and (2) the constant force also opposes the motion ($a < 0$). Beginning with the case where $a > 0$ and using Eq. (13c) for the resistance function, the integrals (14) and (15) become

$$t = \int_{v_0}^{v} \frac{dv}{a - \beta v^2} \quad \cdots \quad (14b) \qquad \text{and} \qquad x = \int_{v_0}^{v} \frac{v\,dv}{a - \beta v^2} \quad \cdots \quad (15b)$$

and, from Eq. (c), we have for the terminal velocity $V = \sqrt{a/\beta}$. When this notation is introduced, the above integrals become

$$t = \frac{V^2}{a} \int_{v_0}^{v} \frac{dv}{V^2 - v^2} \qquad \text{and} \qquad x = \frac{V^2}{a} \int_{v_0}^{v} \frac{v\,dv}{V^2 - v^2}.$$

Integrating, we get

$$t = \frac{V}{a} \tanh^{-1}\left(\frac{v}{V}\right)_{v_0}^{v} = \frac{V}{a}\left[\tanh^{-1}\left(\frac{v}{V}\right) - \tanh^{-1}\left(\frac{v_0}{V}\right)\right], \tag{k}$$

$$x = -\frac{V^2}{2a} \log\,[v^2 - V^2]_{v_0}^{v} = \frac{V^2}{2a} \log \frac{V^2 - v_0^2}{V^2 - v^2}. \tag{l}$$

If the body starts from rest ($v_0 = 0$), these equations reduce to

$$\left.\begin{aligned} v &= V \tanh\left(\frac{at}{V}\right), \\ x &= \frac{V^2}{2a} \log \frac{V^2}{V^2 - v^2}. \end{aligned}\right\} \tag{m}$$

Then eliminating v, we get the displacement-time equation

$$x = \frac{V^2}{a} \log \cosh\left(\frac{at}{V}\right). \tag{n}$$

If the constant force opposes the motion ($a < 0$), the integrals (14) and (15) become

$$t = -\int_{v_0}^{v} \frac{dv}{a + \beta v^2} \tag{14c}$$

and

$$x = -\int_{v_0}^{v} \frac{v\,dv}{a + \beta v^2}, \tag{15c}$$

which, with the notation $V = \sqrt{a/\beta}$, become

$$t = \frac{V^2}{a}\int_{v}^{v_0} \frac{dv}{V^2 + v^2} \qquad \text{and} \qquad x = \frac{V^2}{a}\int_{v}^{v_0} \frac{v\,dv}{V^2 + v^2}.$$

Evaluation of these integrals gives

$$\left.\begin{aligned} t &= \frac{V}{a}\left[\tan^{-1}\left(\frac{v_0}{V}\right) - \tan^{-1}\left(\frac{v}{V}\right)\right], \\ x &= \frac{V^2}{2a}\log\frac{V^2 + v_0^2}{V^2 + v^2}. \end{aligned}\right\} \tag{o}$$

These equations are valid, of course, only as long as v is positive. When the body comes to rest ($v = 0$), we have

$$t = \frac{V}{a}\tan^{-1}\left(\frac{v_0}{V}\right) \tag{p}$$

and

$$x = \frac{V^2}{2a}\log\left(1 + \frac{v_0^2}{V^2}\right). \tag{q}$$

Example: If we take $a = g$, Eqs. (k) to (q) apply to the motion of a projectile fired vertically upward with initial velocity v_0 less than the velocity of sound. Assuming an arrow 0.02 ft. in diameter and 235 grains in weight and taking the coefficient of resistance $C = 1.00$, we have, for air density $\rho = 0.00237$ slug per cu. ft.,

$$\beta = \frac{Ck}{m} = 35.7(10)^{-5}\text{ft.}^{-1}$$

and $a = g = 32.2$ ft. per sec. per sec. Hence the terminal velocity

$$V = \sqrt{\frac{a}{\beta}} = \sqrt{\frac{32.2}{35.7}(10)^5} = 300 \text{ f.p.s.}$$

A 35-lb.-draw bow will give the arrow an initial velocity $v_0 \approx 250$ f.p.s. Then by Eq. (q), the maximum height h that the arrow will attain is

$$h = \frac{V^2}{2g}\log\left(1 + \frac{v_0^2}{V^2}\right) = 1{,}400\log(1.695) = 740 \text{ ft.}$$

We see that this is about three-quarters of the height that would be attained without air resistance. From Eq. (p), the time required to make this ascent is

$$t = \frac{V}{g}\tan^{-1}\left(\frac{v_0}{V}\right) = 9.32\tan^{-1}(0.834) = 6.48 \text{ sec.}$$

For the descent of the arrow, we use Eqs. (m) and (n). Substituting $h = 740$ ft. for x in Eq. (n), we find that

$$\log \cosh\left(\frac{gt'}{V}\right) = 0.2643,$$

from which the time of descent $t' = 7.08$ sec. Substituting this value of t in the first of Eqs. (m), we find, for the velocity with which the arrow returns to us,

$$v = 300 \tanh\left(\frac{g}{V}t'\right) = 300 \tanh(0.760) = 192 \text{ f.p.s.}$$

In all cases where the coefficient of resistance C is not constant, we return to Eqs. (14) and (15), where the functions $1/[\alpha - \beta f(v)]$ or $v/[\alpha - \beta f(v)]$ can be represented, in any given case, by a curve and the integrals evaluated by graphical or numerical integration. Thus, for example, having the resistance curve such as those in Fig. 20 for a ship, we can easily find by graphical integration the time and distance occupied in attaining full speed. Also, the case of a projectile fired with initial velocity $v_0 > c$ can be treated in a similar manner by using a curve like that in Fig. 19.

PROBLEMS

11. Integrate the differential equation $dv/dt = -\beta v^2$, and show that when quadratic resistance acts alone, the rectilinear motion of a particle having initial velocity v_0 is defined by the equations

$$v = \frac{v_0}{1 + \beta v_0 t} \qquad \text{and} \qquad v = v_0 e^{-\beta x}$$

from which, $x = (1/\beta)\log(1 + \beta v_0 t)$.

12. H.M.S. "Greyhound" (displacement 1,161 tons) cuts off its power at a speed of 8 knots and glides to a stop. Referring to Fig. 20a, we find that at 8 knots, $R = 5{,}400$ lb., while at 5 knots, $R = 2{,}000$ lb. Assuming that the resistance is quadratic between 8 and 5 knots and linear below 5 knots, compute the total distance used up in coming to a stop. One knot = 1.689 f.p.s. *Ans.* $x = 3{,}676$ ft.

13. A ship going at full speed V suddenly reverses its engines. Assuming quadratic resistance at all times, find the time and distance occupied in coming to a full stop. [Use Eqs. (p) and (q), page 31.] *Ans.* $t = \pi V/4\alpha$; $x = (V^2/2\alpha)\log 2$.

14. If, in the preceding problem, the displacement of the ship is 1,200 tons and the full speed V is 12 knots, at which speed the total resistance is 25,000 lb., what are the numerical values of t and x above? *Ans.* $t = 47.5$ sec.; $x = 425$ ft.

15. A flyer and his parachute together weigh 220 lb. The parachute [considered as a hemispherical shell ($C = 1.33$)] has a diameter $d = 16$ ft. What is the terminal velocity in air (mass density $\rho = 0.00237$ slug per cu. ft.)?

Ans. $V = \sqrt{\alpha/\beta} = 26.3$ f.p.s.

16. Using the resistance curve in Fig. 20a, determine, by graphical integration, the distance occupied by H.M.S. "Greyhound" in coming to rest if the power is cut off at a speed of 12 knots. The displacement of the ship was 1,161 tons.

17. Assuming the resistance curve in Fig. 19 to apply to a 35-grain .22-caliber rifle bullet fired vertically upward with muzzle velocity $v_0 = 2{,}200$ f.p.s., find the maximum height h to which it will ascend (air density $\rho = 0.00237$ slug per cu. ft.).

18. Assuming the curve in Fig. 19 to apply to a 150-grain .30-caliber rifle bullet fired with muzzle velocity $v_0 = 2{,}800$ f.p.s., find the velocity with which the bullet hits its target at a range of 1,000 yd. Treat the trajectory as a horizontal straight line.

5. Free Vibrations with Viscous Damping.—In preceding articles, we have discussed integration of the differential equation of motion (1), always using such expressions for the force X that the integration could be made by quadrature. In those dynamical problems where the acting force depends upon more than one quantity such as displacement, velocity, or time, the variables are usually inseparable and the problem cannot be reduced to quadratures. As an example, we consider a case where the acting force is a function of both displacement and velocity.

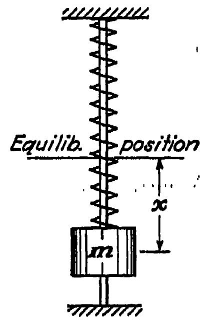


FIG. 22.

Referring to Fig. 22, let us consider a spring-suspended mass m[1] free to vibrate vertically in a medium that offers a viscous resistance, *i.e.*, a resisting force proportional to the first power of the velocity. Such resistance is usually termed *viscous damping*.[2] In this case, if the mass m be displaced any distance x (within the elastic range of the spring) from its position of equilibrium and then released, it will begin to move under the action of a resultant force

$$X = -kx - c\dot{x}, \tag{a}$$

where the spring constant k represents the force necessary to produce unit extension of the spring and the coefficient of damping c represents the magnitude of the viscous resistance at unit velocity. For such forces, the equation of motion (1) becomes

$$m\ddot{x} = -kx - c\dot{x}.$$

Dividing through by m and introducing the notations

$$\frac{k}{m} = p^2 \qquad \text{and} \qquad \frac{c}{m} = 2n, \tag{b}$$

[1] We assume that the mass of the spring itself is small compared with the suspended mass, so that we are dealing with the motion of a single particle.

[2] By using a dashpot, damping can always be arranged to obey this law. Accidental damping such as air resistance is often assumed to do so, although from our discussions in Art. 4, we can expect that in this case, a resistance proportional to the square of the velocity would be more appropriate. Such quadratic damping as well as various combinations of linear and quadratic resistance have been considered by W. E. Milne. See Damped Vibrations: General Theory together with Solutions of Important Special Cases, *Univ. Oregon Pub.*, vol. 2., No. 2., August, 1923; also, Tables of Damped Vibrations, *Univ. Oregon Pub., Math. Ser.*, vol. 1, No. 1, March, 1929.

the equation may be written in the form

$$\ddot{x} + 2n\dot{x} + p^2x = 0. \tag{16}$$

Thus, for *free vibrations with viscous damping*, we obtain a *linear differential equation with constant coefficients.*

As a trial solution of Eq. (16), we take

$$x = Ce^{rt}, \tag{c}$$

where r is a constant. Substituting this trial function for x in Eq. (16), we obtain the *auxiliary equation*

$$r^2 + 2nr + p^2 = 0. \tag{d}$$

This equation determines two values of r for which expression (*c*) can satisfy Eq. (16). They are

$$\left.\begin{aligned} r_1 &= -n + \sqrt{n^2 - p^2}, \\ r_2 &= -n - \sqrt{n^2 - p^2}. \end{aligned}\right\} \tag{e}$$

Hence, we may take the general solution of Eq. (16) as follows:

$$x = Ae^{r_1t} + Be^{r_2t}, \tag{17}$$

where A and B are arbitrary constants of integration.

To attach any physical significance to this solution, we must distinguish between two distinct cases, depending upon whether the radical $\sqrt{n^2 - p^2}$ in expressions (*e*) is real or imaginary, *i.e.*, on whether $n > p$ or $n < p$. Going back to notations (*b*), we see that this rests on the relative magnitudes of the damping coefficient c and the spring constant k. Generally speaking, a large coefficient of damping and a small spring constant will result in real values of r_1 and r_2, while, for the reverse of these conditions, r_1 and r_2 will be complex numbers.

Case I, ($n > p$). In this case, r_1 and r_2 have real values. To evaluate the constants A and B in the general solution (17), we must know the initial conditions of the motion. Assuming, as a particular case, that

$$x = x_0, \qquad \dot{x} = 0, \qquad \text{when } t = 0, \tag{f}$$

we find by direct substitution into Eq. (17) and its first derivative with respect to time that

$$A = -\frac{r_2x_0}{r_1 - r_2}, \qquad B = +\frac{r_1x_0}{r_1 - r_2}. \tag{g}$$

For these values, solution (17) becomes

$$x = \frac{x_0}{r_1 - r_2}(r_1e^{r_2t} - r_2e^{r_1t}). \tag{17a}$$

In connection with this solution, it should be noted that both r_1 and r_2 are *negative*, r_2 being numerically the larger. Thus the displacement x has the same sign as x_0 and approaches zero as a limit when t becomes infinitely large. The displacement-time diagram plotted from Eq. (17*a*) is shown in Fig. 23. We see that the motion is not a vibration at all but simply one in which the suspended mass, after its initial displacement, gradually creeps back toward the equilibrium position but takes theoretically infinite time to get there. This is called *aperiodic motion.* It results from the fact that the damping coefficient is too large compared with the spring constant and is sometimes of practical interest in connection with certain types of electrical measuring instruments such as the galvanometer. In the case where $n = p$, we also obtain aperiodic motion, and the corresponding value of the damping coefficient

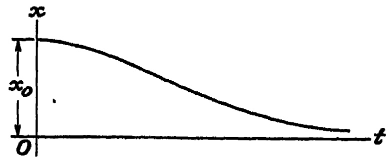


FIG. 23.

$$c = 2nm = 2\sqrt{km} \tag{h}$$

is called *critical damping.*

Case II, ($n < p$). More often, we encounter the case where $n < p$ so that the roots r_1 and r_2 are complex. In discussing this case, it will be convenient to change the form of Eq. (17) in order to see more clearly its physical significance. Remembering that we are now concerned only with the case where $n < p$, we let

$$p^2 - n^2 = p_1^2. \tag{i}$$

Then expressions (*e*) become

$$\left.\begin{aligned} r_1 &= -n + ip_1, \\ r_2 &= -n - ip_1, \end{aligned}\right\} \tag{j}$$

where $i = \sqrt{-1}$, and Eq. (17) may be written in the form

$$x = e^{-nt}(Ae^{ip_1t} + Be^{-ip_1t}). \tag{17b}$$

Using the known relations

$$\left.\begin{aligned} e^{iz} &= \cos z + i \sin z, \\ e^{-iz} &= \cos z - i \sin z, \end{aligned}\right\} \tag{k}$$

Eq. (17*b*), in turn, may be written in the form

$$x = e^{-nt}(C_1\cos p_1t + C_2\sin p_1t), \tag{18}$$

where C_1 and C_2 are new arbitrary constants. To evaluate these constants, we assume now the initial conditions

$$x = x_0, \qquad \dot{x} = \dot{x}_0, \qquad \text{when } t = 0. \tag{l}$$

Then substituting conditions (l) into Eq. (18), together with its first derivative with respect to time, we find

$$C_1 = x_0 \qquad \text{and} \qquad C_2 = \frac{\dot{x}_0}{p_1} + \frac{nx_0}{p_1}, \tag{m}$$

and our solution becomes

$$x = e^{-nt}\left[x_0\cos p_1t + \left(\frac{\dot{x}_0}{p_1} + \frac{nx_0}{p_1}\right)\sin p_1t\right]. \tag{18a}$$

The general solution (18) can be represented in still another form by the following considerations: Referring to Fig. 24, we lay out from the origin O a vector $\overline{OB} = C_1$ that makes with the x-axis the angle p_1t as shown. Then the projection of this vector on the x-axis is $C_1\cos p_1t$ and represents the first term in the parentheses of Eq. (18). Now from the end B of this vector, we lay out $\overline{BC} = C_2$ at right angles to $\overline{OB}$. In this way, the projection of $\overline{BC}$ on the x-axis will be $C_2\sin p_1t$, and we have the second term in the parentheses of Eq. (18). It is now clear from the figure that the sum of the projections of the vectors $\overline{OB}$ and $\overline{BC}$ on the x-axis is the same as the projection of their resultant $\overline{OC}$. Hence, instead of Eq. (18), we may write

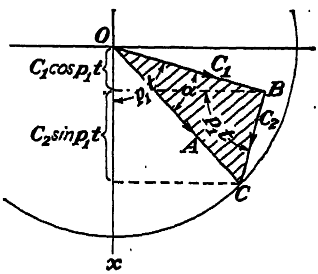


FIG. 24.

$$x = Ae^{-nt}\cos(p_1t - \alpha), \tag{19}$$

wherein

$$\left.\begin{aligned} A &= \sqrt{C_1^2 + C_2^2}, \\ \alpha &= \tan^{-1}\frac{C_2}{C_1} \end{aligned}\right\} \tag{n}$$

are new forms of the arbitrary constants. Using the values of C_1 and C_2 from expressions (m), these new constants become

$$A = \sqrt{x_0^2 + \left(\frac{\dot{x}_0}{p_1} + \frac{nx_0}{p_1}\right)^2} \tag{o}$$

and

$$\alpha = \tan^{-1}\left(\frac{\dot{x}_0}{x_0p_1} + \frac{n}{p_1}\right). \tag{p}$$

A displacement-time curve, plotted from Eq. (19) is shown in Fig. 25, and we see that the motion is now vibratory in nature and represents so-called *damped free vibrations.* Each time that $\cos(p_1t - \alpha)$ becomes equal to ± 1, this curve is tangent to one of the envelopes $x = \pm Ae^{-nt}$. This quantity is called the *amplitude of vibration.* We see that, owing to

damping, the amplitude gradually diminishes with time and that the rate of decay depends on the factor n. When $n = 0$ (no damping), the amplitude remains constant as given by Eq. (o) when $n = 0$. The time $\tau = 2\pi/p_1$ required to complete one cycle of the motion is called the *period of vibration*, and its reciprocal $f = p_1/2\pi$ is called the *frequency of vibration*. We see from Eq. (i) that these expressions also involve the quantity n. The quantity α, as defined by Eq. (p), is called the *phase*

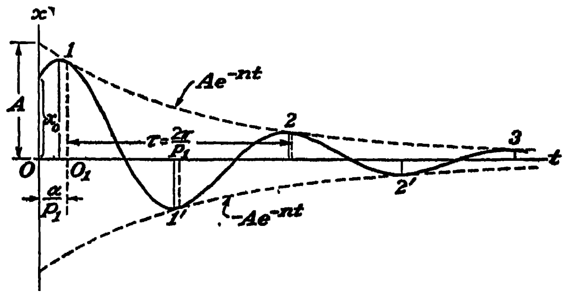


FIG. 25.

angle. We see from the curve in Fig. 25 that it determines the lapse of time between the initial instant $t = 0$ and the first point of tangency with the envelopes $x = \pm Ae^{-nt}$. From these observations, we find that free vibrations of a spring-suspended mass are completely defined by their *amplitude, period,* and *phase* angle. Each of these quantities will now be discussed in more detail.

We begin with a consideration of the amplitude of vibration as represented graphically by the exponential curves $x = \pm Ae^{-nt}$ in Fig. 25. We see that the displacement-time curve is tangent to these curves at the ordinates $t = \alpha/p_1$, $t = (\alpha + \pi)/p_1$, $t = (\alpha + 2\pi)/p_1$, etc. It will be noted that these points of tangency do not coincide exactly with the points of the curve representing extreme positions of the vibrating mass and that, owing to damping, the time interval occupied by the body in moving from an extreme position to a subsequent middle position is greater than that occupied in moving from a middle position to the next extreme position. The rate at which the amplitude diminishes depends upon the quantity n and can be calculated in the following manner: Let x_1 be the displacement at the first point of tangency with the exponential curve and x_s the displacement after $(s - 1)$ complete cycles. Then it is evident from Eq. (19) that

$$\frac{x_1}{x_s} = e^{(s-1)n\tau} \qquad \text{or} \qquad \log\left(\frac{x_1}{x_s}\right) = (s-1)n\tau. \tag{q}$$

We see that the amplitudes at the ends of successive cycles diminish

as a geometric progression. The quantity

$$n\tau = \frac{2\pi n}{\sqrt{p^2 - n^2}}, \tag{r}$$

on which the rate of decay depends, is called the *logarithmic decrement* of the amplitude. It can be determined experimentally by observing in what proportion the amplitude is diminished after an arbitrary number of cycles and then using Eq. (q).

Using notation (i), we see that the period of damped free vibration is

$$\tau = \frac{2\pi}{p_1} = \frac{2\pi}{\sqrt{p^2 - n^2}}. \tag{s}$$

As the value of n varies from 0 (no damping) to p (critical damping), the period τ varies from $2\pi/p$ to ∞, while the frequency of vibration

$$f = \frac{1}{\tau} = \frac{p_1}{2\pi} = \frac{\sqrt{p^2 - n^2}}{2\pi} \tag{t}$$

varies from $p/2\pi$ to 0. Denoting the undamped frequency $p/2\pi$ by f_0, Eq. (t) can be expressed in the form

$$\frac{f^2}{f_0^2} + \frac{n^2}{p^2} = 1. \tag{u}$$

We see that the relation between frequency f and damping n is represented graphically by the quadrant of an ellipse as shown in Fig. 26. For small values of n/p, which we usually have, Eq. (u) can be expressed in the approximate form

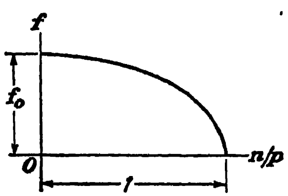


FIG. 26.

$$f \approx f_0\left(1 - \frac{n^2}{2p^2}\right), \tag{v}$$

and we see that the frequency f differs from f_0 only by a small quantity of second order. For example, if $n/p < \frac{1}{10}$, f differs from f_0 by less than $\frac{1}{2}$ per cent. In cases where damping is due principally to air resistance, n will be very small compared with p and the effect of damping on the period or frequency of vibration can be ignored. Consequently, in most practical cases, we have

$$\tau \approx \frac{2\pi}{p} = 2\pi\sqrt{\frac{m}{k}} = 2\pi\sqrt{\frac{\delta_{st}}{g}}, \tag{w}$$

where $\delta_{st} = mg/k$ represents the static elongation induced in the spring by the suspended mass. We note that this period is completely independent of the initial displacement x_0 and initial velocity $\dot{x}_0$ and depends

only on physical constants of the system; it will be referred to as the *natural period* of the system.

The phase angle α, as defined by expression (p), is of less practical interest than either amplitude or period. As we see from expression (p), we can always choose such a relation between initial displacement x_0 and initial velocity $\dot{x}_0$ as to make $\alpha = 0$. This amounts simply to shifting the time origin in Fig. 25 from O to O_1.

PROBLEMS

19. A spring-suspended weight $W = 2$ lb. vibrates with an observed period $\tau = \frac{1}{2}$ sec., and damping is such that after 10 complete cycles, the amplitude has decreased from $x_1 = 2$ in. to $x_{11} = 1$ in. Calculate the coefficient of damping c.

Ans. $c = 0.001436$ lb. per in. per sec.

20. By how much is the natural frequency of a system reduced if the coefficient of damping $c = \frac{1}{2}c_{cr}$?

Ans. $f = \frac{7}{8}f_0$.

6. Forced Vibrations: Harmonic Disturbing Force.—In the preceding article, we have discussed *free vibrations* of a spring-suspended mass with viscous damping. We shall now consider the case where, in addition to a spring force $-kx$ and a resisting force $-c\dot{x}$, there is also an external force $Q = f(t)$. Such a force, the action of which is independent of the motion itself, is called a *disturbing force*. A particular case of practical interest is the periodic disturbing force

$$Q = Q_0 \cos \omega t. \qquad (a)$$

If a spring-suspended electric motor, constrained to move vertically as shown in Fig. 27, has an unbalanced armature that rotates with uniform angular speed ω rad. per sec., a centrifugal force Q_0 is set up and the projection of this force on the vertical x-axis is given by Eq. (a).[1] Under such conditions, the equation of motion (1) for the suspended mass becomes

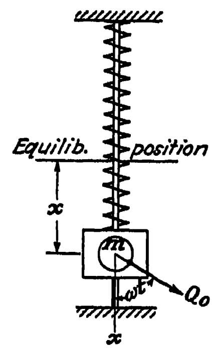


FIG. 27.

$$m\ddot{x} = -kx - c\dot{x} + Q_0 \cos \omega t. \qquad (b)$$

Dividing through by m and introducing the notations

$$p^2 = \frac{k}{m}, \qquad 2n = \frac{c}{m}, \qquad q_0 = \frac{Q_0}{m}, \qquad (c)$$

we obtain

$$\ddot{x} + 2n\dot{x} + p^2 x = q_0 \cos \omega t. \qquad (20)$$

This is the differential equation of *forced vibrations with viscous damping*.

[1] We neglect the effect of vertical motion on the magnitude of the centrifugal force.

A *particular solution* of Eq. (20) can be taken in the form

$$x = A\cos\omega t + B\sin\omega t, \tag{21}$$

where A and B are constants. To determine these constants, we sub-substitute the trial solution (21) into Eq. (20) and obtain

$$(-\omega^2 A + 2n\omega B + p^2 A - q_0)\cos\omega t + (-\omega^2 B - 2n\omega A + p^2 B)\sin\omega t = 0.$$

This equation can be satisfied for all values of t only if the expressions in the parentheses vanish. Thus for calculating A and B, we have the equations

$$-\omega^2 A + 2n\omega B + p^2 A = q_0,$$
$$-\omega^2 B - 2n\omega A + p^2 B = 0,$$

from which

$$\left.\begin{aligned} A &= \frac{q_0(p^2 - \omega^2)}{(p^2 - \omega^2)^2 + 4n^2\omega^2}, \\ B &= \frac{2q_0\omega n}{(p^2 - \omega^2)^2 + 4n^2\omega^2}. \end{aligned}\right\} \tag{d}$$

Substituting these constants into Eq. (21), we obtain the particular solution of Eq. (20).

The *general solution* of Eq. (20) is obtained by adding the particular solution (21) to the general solution of Eq. (16) on page 34, in which the right-hand member is zero. Thus, considering only subcritical damping, we have

$$x = e^{-nt}(C_1\cos p_1 t + C_2\sin p_1 t) + A\cos\omega t + B\sin\omega t. \tag{22}$$

This general solution represents the superposition of *damped free vibrations*, as represented by the first two terms, and *damped forced vibrations*, as represented by the last two terms. The free vibrations have the period

$$\tau = \frac{2\pi}{p_1},$$

as discussed in the preceding article, and the forced vibrations have the period

$$\tau_1 = \frac{2\pi}{\omega}$$

identical with the period of the disturbing force that produces them. Owing to the presence of the factor e^{-nt}, we see that owing to damping, the free vibrations gradually subside, leaving only the steady forced vibrations represented by the last two terms, *i.e.*, by the particular solution (21). These forced vibrations are maintained indefinitely by

the action of the disturbing force and therefore are of great practical importance.

Sometimes, the temporary state of combined free and forced vibrations is of practical interest; and before leaving the general solution (22), we shall discuss briefly one particular case. Neglecting the effect of damping, *i.e.*, taking $n = 0$, expressions (d) for the constants A and B reduce to

$$A = \frac{q_0}{p^2 - \omega^2} \qquad \text{and} \qquad B = 0 \tag{e}$$

and solution (22) becomes

$$x = C_1 \cos pt + C_2 \sin pt + \frac{q_0}{p^2 - \omega^2} \cos \omega t. \tag{23}$$

Assuming now, as a particular case, the initial conditions

$$x = 0, \qquad \dot{x} = 0, \qquad \text{when } t = 0,$$

we find from Eq. (23) and its first derivative with respect to time that

$$C_1 = -\frac{q_0}{p^2 - \omega^2}, \qquad C_2 = 0, \tag{f}$$

and our solution becomes

$$x = \frac{q_0}{p^2 - \omega^2} (\cos \omega t - \cos pt). \tag{23a}$$

From this expression, we see that at the beginning of motion, the disturbing force sets up both free and forced vibrations having the same amplitude. When ω is nearly equal to p, this superposition of two simple harmonic motions having almost, but not quite, the same period results in an interesting phenomenon known as *beating*. To show this analytically, we assume $\omega < p$ and take

$$p - \omega = \delta. \tag{g}$$

Then using the trigonometric identity

$$\cos x - \cos y = -2 \sin \frac{x + y}{2} \sin \frac{x - y}{2},$$

we write Eq. (23a) in the form

$$x = \left(\frac{2q_0}{p^2 - \omega^2} \sin \frac{\delta t}{2}\right) \sin \left(\frac{\omega + p}{2}\right) t. \tag{23b}$$

This expression represents a vibratory motion having the period

$$\tau = \frac{4\pi}{\omega + p} \tag{h}$$

and an amplitude $2q_0 \sin (\delta t/2)/(p^2 - \omega^2)$ that also varies periodically

with time, becoming zero at multiples of the interval

$$T = \frac{2\pi}{\delta} = \frac{2\pi}{p - \omega}. \tag{i}$$

This interval, during which the amplitude rises to a maximum and again falls to zero, is called the *period of beating*. The phenomenon is represented by the displacement-time diagram in Fig. 28, which has been plotted from Eq. (23*b*). Such beats are readily observable, especially

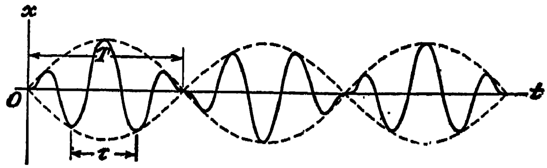


FIG. 28.

when the period of beating T is several times larger than the period of vibration τ. Since damping soon destroys the free vibrations, the phenomenon can be seen only at the beginning of motion.

It will be noted from expression (*i*) that as the frequency of the disturbing force $\omega/2\pi$ approaches closer and closer to the natural frequency $p/2\pi$, the period of beating becomes longer and longer until, when $\omega = p$, it becomes infinite and, without damping, the amplitude increases indefinitely with time. To show this more clearly, we write Eq. (23*b*) as follows:

$$x = \left[\frac{2q_0 \sin\,(\delta t/2)}{(p+\omega)(p-\omega)}\right] \sin\left(\frac{\omega + p}{2}\right) t. \tag{23c}$$

Then as ω approaches the value of p, we have

$$p + \omega \approx 2\omega, \qquad p - \omega = \delta, \qquad \sin\frac{\delta t}{2} \approx \frac{\delta t}{2}, \tag{j}$$

and Eq. (23*c*) becomes

$$x = \frac{q_0 t}{2\omega} \sin \omega t. \tag{23d}$$

Thus, for the condition $\omega = p$, known as *resonance*, the amplitude increases uniformly with time and without limit as shown in Fig. 29. In drawing such a conclusion, it must be remembered that our solution [Eq. (23)] neglects the effect of damping. As a result of damping, the vibrations at resonance do not grow indefinitely but soon reach a steady state in which the amplitude remains constant.

To see this, we return now to the *steady forced vibrations* represented by the particular solution (21). Using the transformation illustrated

in Fig. 24 (see page 36), we may write Eq. (21) in the new form

$$x = C \cos(\omega t - \alpha), \tag{24}$$

where

$$C = \sqrt{A^2 + B^2} = \frac{q_0/p^2}{\sqrt{\left(1 - \frac{\omega^2}{p^2}\right)^2 + \frac{4n^2\omega^2}{p^4}}} \tag{k}$$

and

$$\alpha = \tan^{-1}\left(\frac{B}{A}\right) = \tan^{-1}\left(\frac{2\omega n}{p^2 - \omega^2}\right). \tag{l}$$

From Eq. (24), we see that steady forced vibration with viscous damping is a simple harmonic motion having constant amplitude C, as given by expression (k), *phase angle* α, as given by expression (l), and period $2\pi/\omega$, which is the same as that of the disturbing force regardless of the natural period of the system and regardless of the amount of damping.

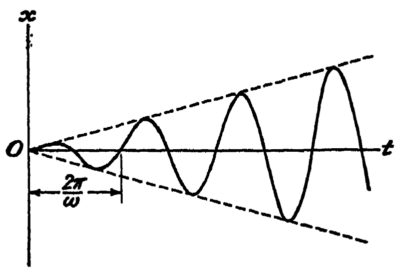


FIG. 29.

Using the values of p^2 and q_0 from notations (c) and introducing the new notation

$$\gamma = \frac{2n}{p}, \tag{m}$$

expression (k) for the amplitude of forced vibration may be written in the more convenient form

$$C = \frac{Q_0}{k}\left[\frac{1}{\sqrt{\left(1 - \frac{\omega^2}{p^2}\right)^2 + \gamma^2 \frac{\omega^2}{p^2}}}\right]. \tag{25}$$

Likewise, expression (l) for the phase angle becomes

$$\alpha = \tan^{-1}\left[\frac{\gamma\omega}{1 - (\omega^2/p^2)}\right]. \tag{26}$$

From Eq. (25), we see that the amplitude of forced vibration is obtained by multiplying the static effect Q_0/k of the disturbing force by the quantity within the brackets. The absolute value of this quantity, which we shall denote by β, is called the *magnification factor*. We see that it depends on the amount of damping, as represented by γ, and also on the ratio ω/p, *i.e.*, on the ratio of the frequency of the disturbing force to the frequency of free vibration of the system. In our further dis-

cussion, we shall refer to these two frequencies, respectively, as the *impressed frequency* $f_1 = \omega/2\pi$ and the *natural frequency* $f_0 = p/2\pi$.*

In Fig. 30, the values of the magnification factor β for various values of γ are plotted against the ratio $\omega/p = f_1/f_0$. From these curves, we see that when the impressed frequency f_1 is small compared with the natural frequency f_0, the value of the magnification factor is not greatly different from unity. This means that during vibration, the displacements x of the suspended mass are approximately those which would be produced by the purely statical action of the disturbing force $Q_0 \cos \omega t$.

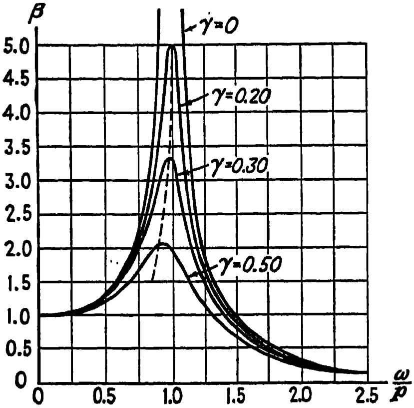


FIG. 30.

When ω is large compared with p, *i.e.*, when the impressed frequency is much greater than the natural frequency, the value of the magnification factor tends toward zero, regardless of the amount of damping. This means that a high-frequency disturbing force produces practically no forced vibrations of a system that has a low natural frequency.

In both extreme cases ($\omega \ll p$ and $\omega \gg p$), we note that damping has only a secondary effect on the magnitude of the magnification factor β. Thus, in these extreme cases, it is justifiable in discussing forced vibrations to neglect the effect of damping entirely, in which case Eq. (24) takes the much simpler form

$$x = \frac{Q_0}{k} \frac{1}{1 - (\omega^2/p^2)} \cos \omega t. \tag{27}$$

From this expression, we see that below resonance ($\omega \ll p$), the forced vibrations are *in phase* with the disturbing force while above resonance ($\omega \gg p$) they are *out of phase*. In the latter case, this means that while Q_0 acts downward in Fig. 27, the suspended mass m is moving upward and vice versa.

As the value of ω approaches that of p, *i.e.*, as the impressed frequency f_1 approaches the natural frequency f_0, the magnification factor grows rapidly and, as we see from the figure, its value is very sensitive to changes in the amount of damping, as represented by γ. It will also be noted that the maximum value of β occurs for a value of ω/p slightly less

* Sometimes, to avoid writing $\omega/2\pi$ and $p/2\pi$, we refer to ω and p, respectively, as impressed and natural *angular frequencies*.

than unity, *i.e.*, slightly below resonance. Setting the derivative of β with respect to ω/p equal to zero, we find that the maximum occurs when

$$\frac{\omega}{p} = \sqrt{1 - \frac{\gamma^2}{2}}. \tag{n}$$

Since γ is usually very small, for which case the maximum β occurs very near to resonance, it is common practice to take the value of β at resonance as the maximum. Then from Eq. (25), the maximum amplitude becomes

$$C_{\max} \approx \frac{Q_0}{k\gamma}. \tag{o}$$

We see from this expression that for small damping, the amplitude of forced vibration can become extremely large in the condition of resonance. Since large amplitudes mean large stresses in the spring, the condition of resonance is usually to be regarded as a dangerous one and is to be avoided whenever possible.

Regarding the response of a spring-suspended mass to the action of a periodic disturbing force for various values of the ratio ω/p, we should like to quote here a brief remark by C. E. Inglis:[1]

> In this behavior of the spring-supported mass, there is something almost human; it objects to being rushed. If coaxed gently and not hurried too much, it responds with perfect docility; but if urged to bestir itself at more than its normal gait, it exhibits a mulish perversity of disposition. Such movement as it makes under this compulsion is always in a retrograde direction, and the more it is rushed the less it condescends to move. On the other hand, if it is stimulated with its own natural inborn frequency, it plays up with an exuberance of spirit which may be very embarrassing.

We turn now to the phase relationship between the forced vibrations and the disturbing force that produces them. This is represented by the phase angle α in Eq. (24), the value of which is given by expression (26). Since the disturbing force varies according to $\cos \omega t$ and the forced vibrations according to $\cos(\omega t - \alpha)$, we say that the angle α represents the *lag* of the vibrations behind the disturbing force. That is, when the force Q_0 in Fig. 27 is directed straight down, the suspended mass on which it acts is not yet in its lowest position but arrives there α/ω sec. later, by which time the force Q_0 has advanced to a position where it makes the angle α with the vertical x-axis. Expression (26) shows that the value of α, like that of β, depends both upon the relative amount of damping γ and upon the ratio ω/p. The curves in Fig. 31 show the variation in the phase angle α with the ratio ω/p for several values of

[1] See C. E. INGLIS, "A Mathematical Treatise on Vibrations in Railway Bridges," p. x, Cambridge, London, 1934.

the damping factor γ. We see that when $\gamma = 0$ (no damping), the forced vibrations are exactly in phase with the disturbing force for all values of $\omega/p < 1$ and a full half cycle out of phase for all values of $\omega/p > 1$. Also for this condition, the phase angle is seen to be indeterminate at resonance ($\omega = p$). These observations coincide completely with those made previously on the basis of Eq. (27).

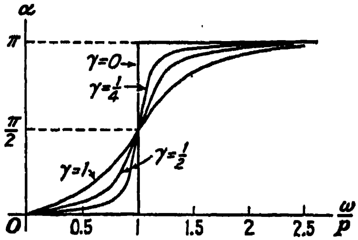


FIG. 31.

When damping is present, we note a continual change in α as the ratio ω/p increases. Also, regardless of the amount of damping, we have $\alpha = \pi/2$ at resonance. That is, at resonance, the forced vibrations lag behind the disturbing force by one quarter cycle. In Fig. 27, for example, the force Q_0 is directed downward when the vibrating mass passes through its middle position; and by the time the mass has moved down to its lowest position, the force Q_0 has rotated through 90 deg. and acts horizontally to the right, etc.

For values of ω/p either well below resonance or well above resonance, we note that a moderate amount of damping, *i.e.*, a small value of γ, has only a secondary effect upon the phase angle α. That is, well below resonance, it is practically zero, while well above resonance, it is practically π. This means, again, that in discussing forced vibrations well away from the condition of resonance, the effects of damping can be ignored.

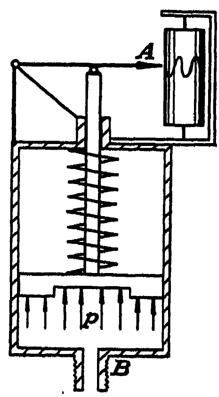


FIG. 32.

Corresponding to each of the three rather distinct regions of the magnification factor diagram of Fig. 30, representing the phenomenon of forced vibrations (1) well below resonance, (2) well above resonance, and (3) near resonance, there are many important technical applications, some of which we shall discuss briefly.

For the condition well below resonance, we note that the chief characteristic of the motion is that the displacements produced by the dynamic action of the disturbing force do not differ much from those which would result from a purely static action of the same force. This fact is useful in the design of *pressure indicators* for measuring variable forces such as steam pressure in the cylinder of an engine. Such an instrument usually consists of a small piston attached to a spring as shown in Fig. 32. The natural frequency of such a system depends upon the spring constant k and the piston mass m. The instrument is con-

nected to the cylinder of an engine at B so that the piston is at all times subjected to the same pressure as that existing in the engine cylinder. Since this pressure varies periodically,[1] it produces forced vibrations of the piston, which are recorded on the uniformly rotating drum by the pencil A. In order that the displacement of the piston be approximately the same as that which any instantaneous static value of the pressure would produce, it is essential that the natural frequency of the indicator system be several times greater than the frequency of fluctuation in the variable pressure to be recorded. Under these conditions, the ratio ω/p will be small and the magnification factor will differ but little from unity. Thus the ordinates of the record can be taken as proportional to the pressure. It is seen that the general requirements for the instrument are a lightweight piston and a stiff spring, since this combination results in a high natural frequency.

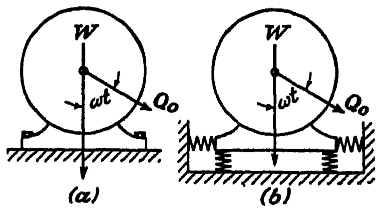


FIG. 33.

We consider next an example of forced vibration well above resonance, where the ratio ω/p is large. This condition, as we have already seen, is characterized by very small amplitudes and a half-cycle phase difference between disturbing force and forced vibration. These facts are utilized in the design of *spring mountings* to be placed between an engine and its foundation. If an electric generator, for example, is rigidly bolted to its foundation (Fig. 33*a*) and there is some unbalance in the rotor, the corresponding centrifugal force Q_0 will be transmitted to the floor and may produce undesirable disturbance and noise. In Fig. 33*b*, the same generator is mounted on flexible springs so that the system has a low natural frequency compared with its operating speed. Under these conditions, the generator practically stands still in space and the forces transmitted through the springs to the foundation are practically constant. The portion of the fluctuating force $Q_0 \cos \omega t$ that gets through to the foundation is only a small fraction of that suffered in Fig. 33*a*.

The principle of isolation illustrated above works equally well in reverse; *i.e.*, if we wish to protect some sensitive instruments from unavoidable building vibrations, we place them on a heavy platform suspended from the ceiling by flexible springs as shown in Fig. 34. If the natural frequency of the platform is small compared with the building's natural frequency, the instruments will suffer only a small fraction of the motion of the ceiling.

[1] We assume a simple harmonic variation.

The system in Fig. 34 requires a few additional remarks. Here, instead of a disturbing force $Q_0 \cos \omega t$, we have a *forced motion*

$$x_1 = a \cos \omega t \tag{p}$$

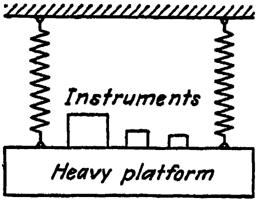


FIG. 34.

of the upper ends of the springs. In such case, the equation of motion for the suspended mass becomes

$$m\ddot{x} = -k(x - x_1) - c\dot{x}, \tag{q}$$

where $(x - x_1)$ represents the net extension of the springs over and above that due to the static action of the gravity force mg. Using the first and second of notations (c), together with expression (p), Eq. (q) becomes

$$\ddot{x} + 2n\dot{x} + p^2x = \frac{ak}{m} \cos \omega t, \tag{r}$$

and we see that if, instead of the last of notations (c), we take

$$q_0 = \frac{ak}{m}, \tag{s}$$

we come to the same differential equation of motion as Eq. (20). Hence all our conclusions about forced vibrations of the system in Fig. 27 apply equally well to a system like that in Fig. 34.

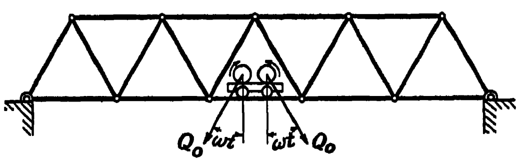


FIG. 35.

A practical application of the resonance phenomenon is illustrated in Fig. 35, where a so-called *vibrator* is used to determine experimentally the natural frequency of vibration of a bridge or other large structure. This machine consists essentially of two identical rotors purposely unbalanced so that when they rotate in opposite directions, they set up two equal centrifugal forces Q_0. Furthermore, they are so adjusted that these forces always make the same angle ωt with the vertical, thus causing their horizontal projections to balance each other, while the vertical projections add together to give a vertical pulsating force, the frequency of which can be controlled by varying the speed of rotation. Suitably mounted on a car, the vibrator can be stationed on the bridge and gradually speeded up until the resonance condition is reached. This condition is easy to recognize owing to the large amplitude of forced

vibration of the bridge. It is necessary then only to note the r.p.s. of the vibrator to have the natural frequency of the bridge.

PROBLEMS

21. The spring-suspended mass in Fig. 27 has a weight $W = 10$ lb. and the spring constant $k = 100$ lb. per in. The coefficient of damping $c = 0.1$ lb. per in. per sec. Construct, to scale, the magnification factor diagram. At what value of ω/p does the magnification factor β have its maximum value, and what is this maximum?

Ans. $\beta_{max} = 16.13$ at $\omega/p = 0.999$.

22. Show that the damping factor γ introduced in notation (m) represents twice the ratio of actual to critical damping; *i.e.*, for critical damping, we have $\gamma = 2$.

23. A steam-pressure indicator like that shown in Fig. 32 uses a spring with constant $k = 100$ lb. per in. Determine the maximum weight W for the piston if the largest frequency of simple harmonic fluctuating pressure to be measured is 600 c.p.m. and a limit of 2 per cent error is imposed. *Ans.* $W \lesseqgtr 0.196$ lb.

24. Determine the spring constant k for the nest of springs in Fig. 33b that will reduce the disturbing force transmitted to the foundation to one-tenth of what it is when the generator is solidly mounted as in Fig. 33a. The generator operates at 1,800 r.p.m. and weighs 200 lb. *Ans.* $k = 1{,}672$ lb. per in.

7. Forced Vibration: General Disturbing Force.—In the preceding article, we considered only the special case of a periodic disturbing force

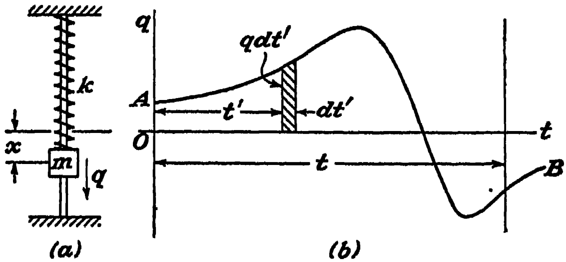


FIG. 36.

$Q = Q_0 \cos \omega t$. While this is the most important case, there are many others for which the differential equation of motion of a spring-suspended mass m (Fig. 36a) may take the more general form

$$\ddot{x} + 2n\dot{x} + p^2x = q = f(t), \tag{28}$$

where $q = Q/m$ is the disturbing force per unit of suspended mass. In dealing with this equation, we shall follow a somewhat different procedure from that used in the previous article. Referring to Fig. 36b, we assume that the disturbing force q per unit of mass is given by a curve AB. Then for any instant t', we consider one elemental impulse $q\,dt'$ as represented by the shaded strip in the diagram. This one impulse imparts to each unit of mass an instantaneous increase in velocity $d\dot{x} = q\,dt'$, regardless of what other forces, such as the spring force, may be acting upon it and regardless of its displacement and velocity at the instant t'. Treating

this increment of velocity as if it were an initial velocity (at the instant t') and using Eq. (18a), page 36, we conclude that the corresponding displacement of the spring-suspended mass at any later time t will be

$$dx = e^{-n(t-t')} \frac{q\,dt'}{p_1} \sin p_1(t - t'). \tag{a}$$

Since each impulse $q\,dt'$ between $t' = 0$ and $t' = t$ has a like effect, we obtain, as a result of the continuous action of the disturbing force q, the total displacement

$$x_1 = \frac{1}{p_1} \int_0^t q e^{-n(t-t')} \sin p_1(t - t')dt'. \tag{29}$$

This expression still does not include the effect of any initial displacement x_0 or initial velocity $\dot{x}_0$, when $t = 0$. These effects, however, are exactly those represented by Eq. (18a); hence, for a complete solution of Eq. (28), we may write

$$x = e^{-nt}\left[x_0 \cos p_1 t + \left(\frac{\dot{x}_0}{p_1} + \frac{n x_0}{p_1}\right) \sin p_1 t\right] + \frac{1}{p_1} \int_0^t q e^{-n(t-t')} \sin p_1(t - t')dt'. \tag{30}$$

At first glance, it might appear that the first term in this expression represented free vibrations and the last term forced vibrations as distinguished in the preceding article. This, however, is not quite the case. Actually, the first term represents the effect of initial displacement and initial velocity only; the last term represents the complete effect of the disturbing force $q = f(t')$. As we already saw in the preceding article (see page 40), the disturbing force produces on its own account both free and forced vibrations and all this together is accounted for by the integral in Eq. (30), *i.e.*, by the solution (29). For this reason, Eqs. (29) and (30) are especially useful in studying the early effects of a disturbing force before damping has had time to dissipate the free vibrations.

If damping is neglected, we have $n = 0$ and $p_1 = p$, for which Eq. (30) reduces to

$$x = x_0 \cos pt + \frac{\dot{x}_0}{p} \sin pt + \frac{1}{p} \int_0^t q \sin p(t - t')dt'. \tag{31}$$

For convenience in further calculations, it will be helpful to transform the integral[1] in Eq. (31) as follows: Using the relation

$$\sin(pt - pt') = \sin pt \cos pt' - \cos pt \sin pt',$$

we write, for that part of the displacement due to the disturbing force

[1] This integral is sometimes called Duhamel's integral after the French mathematician J. M. C. Duhamel.

only,

$$x_1 = \frac{1}{p}\left(\sin pt \int_0^t q \cos pt'dt' - \cos pt \int_0^t q \sin pt'dt'\right) \qquad (b)$$

Then, introducing the notations

$$A = -\frac{1}{p}\int_0^t q \sin pt'dt', \qquad B = \frac{1}{p}\int_0^t q \cos pt'dt', \qquad (32)$$

we have, for a general solution without damping,

$$x = x_0\cos pt + \frac{\dot{x}_0}{p}\sin pt + A \cos pt + B \sin pt. \qquad (33)$$

Thus, without damping, the solution of Eq. (28) reduces to the evaluation of the two integrals A and B as defined by expressions (32).

As a first example, let us reconsider the case already studied in Art. 6, where $q = q_0\cos \omega t'$. In such case, the integrals (32) become

$$A = -\frac{q_0}{p}\int_0^t \sin pt'\cos \omega t'dt' = \frac{q_0}{p}\left[\frac{\cos(p-\omega)t - 1}{2(p-\omega)} + \frac{\cos(p+\omega)t - 1}{2(p+\omega)}\right],$$

$$B = \frac{q_0}{p}\int_0^t \cos pt'\cos \omega t'dt' = \frac{q_0}{p}\left[\frac{\sin(p-\omega)t}{2(p-\omega)} + \frac{\sin(p+\omega)t}{2(p+\omega)}\right].$$

Substituting these values into Eq. (33) and reducing, we obtain

$$x = x_0\cos pt + \frac{\dot{x}_0}{p}\sin pt + \frac{q_0}{p^2-\omega^2}(\cos \omega t - \cos pt). \qquad (33a)$$

Taking $x_0 = \dot{x}_0 = 0$, expression (33a) coincides with expression (23a) on page 41 as it should.

Equation (31) can be used to advantage in studying the motion of a spring-suspended mass under the action of a series of discontinuous impulses. If, owing to such impulses, the mass obtains increments of velocity Δv_0, Δv_1, Δv_2, . . . at the moments $t = 0$, $t = t'$, $t = t''$, . . . , we have, for $x_0 = 0$,

$$x = \frac{1}{p}[\Delta v_0\sin pt + \Delta v_1\sin p(t - t') + \Delta v_2\sin p(t - t'') + \cdot\cdot\cdot]. \qquad (31a)$$

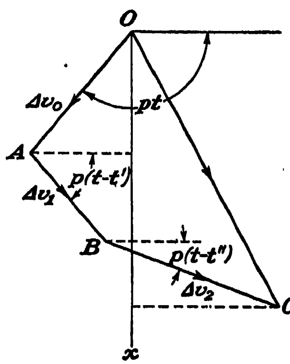


FIG. 37.

We see that this expression is simply the sum of projections on the vertical x-axis of vectors $\overline{\Delta v_0}$, $\overline{\Delta v_1}$, $\overline{\Delta v_2}$, . . . making, with the horizontal, the angles pt, $p(t - t')$, $p(t - t'')$, ... Thus, for any chosen value of t, the summation may be evaluated graphically as shown in Fig. 37. The vertical projection of the

resultant vector $\overline{OC}$, when multiplied by $1/p$, gives the displacement x expressed by Eq. (31a). We note also that the magnitude of $\overline{OC}$ determines the amplitude of the ensuing free vibrations, assuming that no further impulses occur.

Looking again at Eq. (31a), we see that if we take a constant interval of time $\Delta = 2\pi/p = \tau$ between impulses, the terms in the parentheses are all in phase; thus, owing to such a succession of impulses in resonance, the amplitude will be built up without limit. The same result will follow from a succession of alternately positive and negative impulses with the time interval $\Delta = \pi/p = \tau/2$. These observations lead to the important practical conclusion that any periodic disturbing force, in resonance, builds up large amplitudes of forced vibration; it does not need to be simple harmonic in character to do so.

In the case of a constant disturbing force Q suddenly applied to the suspended mass at the initial moment $t = 0$, we have $q = Q/m =$ constant, for which the integrals A and B become

$$A = \frac{q}{p^2}(\cos pt - 1), \qquad B = \frac{q}{p^2}\sin pt. \tag{c}$$

Substituting these values in Eq. (33), we obtain

$$x = x_0 \cos pt + \frac{\dot{x}_0}{p}\sin pt + \frac{q}{p^2}(1 - \cos pt). \tag{33b}$$

Noting that

$$\frac{q}{p^2} = \frac{Q}{k} = x_{st} \tag{d}$$

represents the static displacement corresponding to the constant force Q, we rewrite Eq. (33b) in the form

$$x = x_{st} + (x_0 - x_{st})\cos pt + \frac{\dot{x}_0}{p}\sin pt. \tag{e}$$

From this expression, we see that a suddenly applied constant force produces free vibrations of amplitude

$$C = \sqrt{(x_0 - x_{st})^2 + \left(\frac{\dot{x}_0}{p}\right)^2} \tag{f}$$

superimposed upon a constant displacement $x_{st} = Q/k$. In the particular case where $x_0 = \dot{x}_0 = 0$, expression (f) for amplitude reduces to $C = x_{st}$ and we have $x_{max} = 2x_{st}$ and $x_{min} = 0$. Thus a suddenly applied constant force Q produces a maximum deflection twice as great as the static effect of the same force.

Equation (33b) assumes the constant force Q, once applied, to act indefinitely. If it acts only for an interval of time Δ and then is suddenly

removed, we substitute Δ for t in Eq. (33*b*) and its first derivative with respect to time and find for the displacement and velocity at the end of the interval

$$\left.\begin{aligned} x_\Delta &= x_0 \cos p\Delta + \frac{\dot{x}_0}{p}\sin p\Delta + \frac{q}{p^2}(1 - \cos p\Delta), \\ \dot{x}_\Delta &= -px_0 \sin p\Delta + \dot{x}_0 \cos p\Delta + \frac{q}{p}\sin p\Delta. \end{aligned}\right\} \tag{34}$$

In the particular case where $x_0 = \dot{x}_0 = 0$, these expressions reduce to

$$\left.\begin{aligned} x_\Delta &= \frac{q}{p^2}(1 - \cos p\Delta), \\ \dot{x}_\Delta &= \frac{q}{p}\sin p\Delta. \end{aligned}\right\} \tag{g}$$

If these quantities are treated as initial displacement and initial velocity at the instant Δ, the ensuing free vibrations, from Eq. (33), become

$$x = x_\Delta \cos p(t - \Delta) + \frac{\dot{x}_\Delta}{p}\sin p(t - \Delta), \tag{33c}$$

where $t > \Delta$ is still counted from the original initial moment. These undamped free vibrations have the amplitude

$$a = \sqrt{x_\Delta^2 + \left(\frac{\dot{x}_\Delta}{p}\right)^2},$$

which, by using expressions (*g*), becomes

$$a = \frac{2q}{p^2}\sin\frac{p\Delta}{2} = \frac{Q}{k}\sin\frac{\pi\Delta}{\tau}. \tag{h}$$

We see that this amplitude depends upon the ratio Δ/τ, *i.e.*, upon the ratio of the duration of the constant force to the natural period of the system. By taking $\Delta/\tau = \frac{1}{2}$, we obtain $a = 2Q/k$. By taking $\Delta/\tau = 1$, we obtain $a = 0$; *i.e.*, there will be no vibrations after removal of the constant force Q. In the first case, Q acts through the displacement from 0 to a and does positive work on the system. After removal of the force in the extreme position, the system, without damping, retains this energy and we have free vibrations equivalent to an initial displacement $2Q/k$. In the second case, the constant force does positive work from 0 to a and negative work from a back to 0; the net work becomes zero; and the system acquires no increase in energy. If the suspended mass was initially at rest in its equilibrium position, it must remain at rest in its equilibrium position when the force is taken away.

Equations (34) lend themselves to a simple numerical solution of the

problem of forced vibration of a spring-suspended mass under the action of an irregular disturbing force as represented by the smooth curve in Fig. 38. Replacing this curve by a suitable step curve with the constant time interval Δ, we see that such a disturbing force can be considered as a succession of constant forces Q_1, Q_2, Q_3, . . . , each acting for the time interval Δ and then giving way to the next. Assuming that we know the

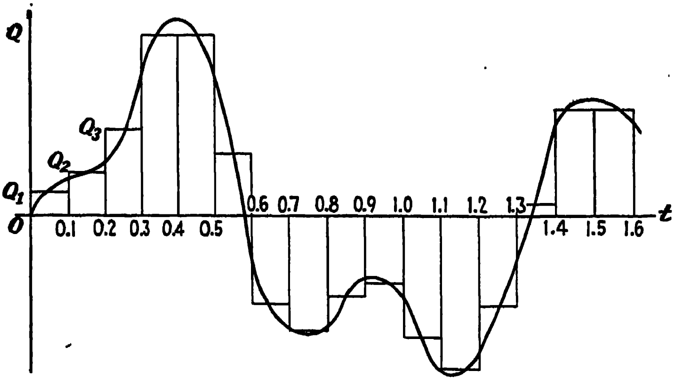


FIG. 38.

initial displacement x_0 and the initial velocity $\dot{x}_0$, we use Eqs. (34) and find for the displacement and velocity at the end of the first interval

$$\left.\begin{aligned} x_1 &= x_0 \cos p\Delta + \frac{\dot{x}_0}{p}\sin p\Delta + \frac{Q_1}{k}(1 - \cos p\Delta), \\ \frac{\dot{x}_1}{p} &= -x_0 \sin p\Delta + \frac{\dot{x}_0}{p}\cos p\Delta + \frac{Q_1}{k}\sin p\Delta. \end{aligned}\right\} \qquad (i)$$

Then taking these values of x_1 and $\dot{x}_1$ as new initial displacement and velocity and using Eqs. (34) again, we have at the end of the second interval

$$\left.\begin{aligned} x_2 &= x_1 \cos p\Delta + \frac{\dot{x}_1}{p}\sin p\Delta + \frac{Q_2}{k}(1 - \cos p\Delta), \\ \frac{\dot{x}_2}{p} &= -x_1 \sin p\Delta + \frac{\dot{x}_1}{p}\cos p\Delta + \frac{Q_2}{k}\sin p\Delta. \end{aligned}\right\} \qquad (j)$$

Continuing in this way, we obtain a series of values of the displacement x corresponding to the times 0, Δ, 2Δ, 3Δ, . . . from which a displacement-time curve for the motion can easily be constructed. To simplify the calculations, the interval Δ should be chosen so that $p\Delta$ is some multiple of 2π. An example will serve to illustrate the procedure.

Example: We assume that the system in Fig. 36a has a spring constant $k = 10$ lb. per in. and a natural period $\tau = 1.2$ sec. The disturbing force $Q = F(t)$, as shown in Fig. 38, is assumed to be defined by the numerical values of Q recorded in the

TABLE IV

(1) t sec.	(2) Q_i lb.	(3) $x_{st} = \frac{Q_i}{k}$ in.	(4) $x_{st}(1 - \cos p\Delta)$ in.	(5) $x_{st} \sin p\Delta$ in.	(6) x in.	(7) $x \cos p\Delta$ in.	(8) $x \sin p\Delta$ in.	(9) $\frac{\dot{x}}{p}$ in.	(10) $\frac{\dot{x}}{p} \cos p\Delta$ in.	(11) $\frac{\dot{x}}{p} \sin p\Delta$ in.
0	....	......	....	....	0	0	0	0	0	0
0.1	2.50	0.25	0.034	0.125	0.033	0.029	0.017	0.125	0.108	0.062
0.2	4.60	0.46	0.062	0.230	0.153	0.133	0.076	0.322	0.278	0.161
0.3	9.20	0.92	0.123	0.460	0.417	0.360	0.208	0.662	0.572	0.331
0.4	19.00	1.90	0.255	0.950	0.945	0.818	0.472	1.314	1.136	0.657
0.5	19.00	1.90	0.255	0.950	1.729	1.498	0.875	1.614	1.397	0.807
0.6	6.60	0.66	0.088	0.330	2.393	2.070	1.197	0.852	0.737	0.426
0.7	− 9.20	−0.92	−0.123	−0.460	2.373	2.052	1.186	−0.920	−0.796	−0.459
0.8	−12.00	−1.20	−0.161	−0.600	1.432	1.240	0.716	−2.582	−2.232	−1.291
0.9	− 8.40	−0.84	−0.112	−0.420	−0.163	−0.141	−0.081	−3.368	−2.915	−1.684
1.0	− 7.00	−0.70	−0.094	−0.350	−1.919	−1.660	−0.959	−3.184	−2.755	−1.592
1.1	−12.70	−1.27	−0.170	−0.635	−3.422	−2.960	−1.711	−2.431	−2.104	−1.215
1.2	−16.00	−1.60	−0.214	−0.800	−4.389	−3.800	−2.195	−1.193	−1.033	−0.597
1.3	− 9.30	−0.93	−0.124	−0.465	−4.521	−3.914	−2.260	+0.697	+0.604	+0.348
1.4	+ 1.50	+0.15	+0.020	+0.075	−3.546	−3.072	−1.773	+2.939	+2.544	+1.469
1.5	+11.20	+1.12	+0.150	+0.560	−1.453	−1.258	−0.727	+4.877	+4.220	+2.439
1.6	+11.20	+1.12	+0.150	+0.560	+1.331	....	....	+5.497		

second column of Table IV, page 55. We have taken $\Delta = 0.1$ sec.; thus $\Delta/\tau = \frac{1}{12}$ and $p\Delta = \pi/6 = 30$ deg.; $\cos p\Delta = 0.866$, $\sin p\Delta = 0.500$, $1 - \cos p\Delta = 0.134$.

We have also assumed $x_0 = \dot{x}_0 = 0$; *i.e.*, the suspended mass m was at rest in its position of equilibrium at the initial moment $t = 0$. The procedure is perfectly regular. The first five columns of Table IV are filled out completely at the outset, and then the initial values x_0 and $\dot{x}_0/p$ (both zero in our case) are placed in the first line of columns (6) and (9), respectively. Then using Eqs. (*i*), the values of x_1 and $\dot{x}_1/p$ are computed and recorded in the second line of columns (6) and (9), after which the remaining blank spaces in the second line can be filled in. Then using Eqs. (*j*), the values of x_2 and $\dot{x}_2/p$ are computed and recorded in the third line of columns (6) and (9), etc. In Fig. 39, the displacement-time curve for the first $1\frac{1}{2}$ sec. of motion is plotted from the data in columns (1) and (6).

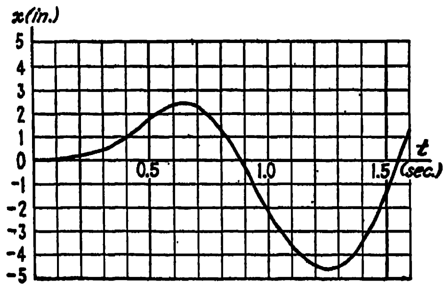


FIG. 39.

Equation (33) also lends itself to a simple graphical treatment.[1] Multiplying and dividing each of the integrals (32) by p and replacing q/p^2 by Q/k, we have

$$A = -\int_0^\phi \frac{Q}{k} \sin \phi \, d\phi, \qquad B = \int_0^\phi \frac{Q}{k} \cos \phi \, d\phi, \tag{k}$$

where in the angle $\phi = pt$ is taken as a new variable. Now referring to Fig. 38, we replace the given curve $Q = F(t)$ by a step curve having over each interval of time Δ a constant value Q_i. If this step curve is used instead of the true curve, the integrals (*k*) may now be replaced by the summations

$$A = \sum \frac{Q_i}{k} \int_{\phi_{i-1}}^{\phi_i} d(\cos \phi), \qquad B = \sum \frac{Q_i}{k} \int_{\phi_{i-1}}^{\phi_i} d(\sin \phi),$$

where $\phi_i = ip\Delta$. Introducing the notation $r_i = Q_i/k$ and integrating, we get

$$\left.\begin{aligned} A &= \Sigma r_i(\cos \phi_i - \cos \phi_{i-1}), \\ B &= \Sigma r_i(\sin \phi_i - \sin \phi_{i-1}), \end{aligned}\right\} \tag{35}$$

[1] The method to be described is due to E. Meissner. See "Graphische Analysis vermittelst des Linienbildes einer Funktion," Verlag der Schweizerischen Bauzeitung, Zurich, 1932.

where r_i is the static displacement corresponding to each value Q_i of the disturbing force.

We now refer to Fig. 40a for a graphical evaluation of A and B. Beginning with a horizontal line $O_1P_0 = r_1$ and with O_1 as a center and r_1 as a radius, we construct a circular arc P_0P_1 whose central angle is $p\Delta$ as shown. This done, we take O_2 on the prolongation of P_1O_1 so that

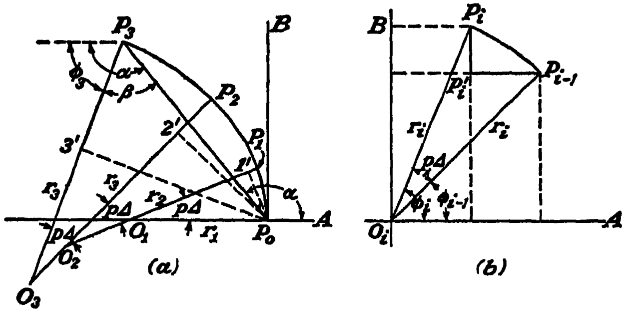


FIG. 40.

$O_2P_1 = r_2$. Then with O_2 as a center and r_2 as a radius, we make the next arc P_1P_2 again with the central angle $p\Delta$. After this, O_3 is taken on the prolongation of P_2O_2; and with O_3 as a center and r_3 as a radius, the arc P_2P_3 is made, etc.

Now in Fig. 40b, we consider the arc $P_{i-1}P_i$ whose center is O_i and whose radius is r_i. The horizontal and vertical projections of this arc, respectively, are seen to be

$$\begin{aligned} P_{i-1}P_i' &= r_i(\cos\phi_i - \cos\phi_{i-1}), \\ P_i'P_i &= r_i(\sin\phi_i - \sin\phi_{i-1}), \end{aligned}$$

which are exactly the quantities under the summation signs in expressions (35). Hence, the horizontal and vertical projections of any point P_i on the curve $P_0P_1P_2 \ldots P_n$ in Fig. 40a determine, respectively, the values of A and B for the corresponding value of the angle $\phi_i = pt$ that the radius O_iP_i makes with the horizontal.

To give further significance to the construction in Fig. 40a, we reconsider Eq. (33), page 51, and write it in the form

$$x = (x_0 + A)\cos pt + \left(\frac{\dot{x}_0}{p} + B\right)\sin pt. \tag{33d}$$

Then using the transformation illustrated in Fig. 24, page 36, we write

$$x = C\cos(pt - \alpha), \tag{36}$$

where

$$C = \sqrt{(x_0 + A)^2 + \left(\frac{\dot{x}_0}{p} + B\right)^2} \tag{l}$$

and

$$\alpha = \tan^{-1}\left(\frac{(\dot{x}_0/p) + B}{x_0 + A}\right). \qquad (m)$$

In the particular case where $x_0 = \dot{x}_0 = 0$, expressions (l) and (m) reduce to

$$C = \sqrt{A^2 + B^2}, \qquad (l')$$

$$\alpha = \tan^{-1}\left(\frac{B}{A}\right). \qquad (m')$$

Now, referring again to Fig. 40a, we see that any radius vector such as $\overline{P_0P_3}$ is equal to the quantity C as defined by Eq. (l') and that the angle α which it makes with the horizontal is that defined by Eq. (m'). Furthermore, since the radius O_3P_3 makes the angle $\phi_3 = 3p\Delta$ with the horizontal, it follows that the angle β between the radius vector $\overline{P_0P_3}$ and the radius O_3P_3 is

$$\beta = -(\phi_3 - \alpha)$$

as shown. Then since $\cos(-\beta) = \cos\beta$, we conclude finally that the projection of $\overline{P_0P_3}$ on the corresponding radius O_3P_3 represents the value of x as given by Eq. (36). Since this reasoning holds for each point P_i on the curve $P_0P_1P_2 \ldots P_n$, the distances P_11', P_22', P_33', . . . give the values of x corresponding to $t = \Delta$, $t = 2\Delta$, $t = 3\Delta$, ...

For the numerical data given in columns (1) and (3) of Table IV, such graphical solution is shown in Fig. 41. The values of x scaled from this construction are shown below in comparison with the computed values obtained in Table IV.

Time t	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8
Graphical x	0	0.03	0.15	0.42	0.95	1.74	2.41	2.39	1.45
Numerical x	0	0.033	0.153	0.417	0.945	1.729	2.393	2.373	1.432

Time t	0.9	1.0	1.1	1.2	1.3	1.4	1.5	1.6
Graphical x	−0.15	−1.92	−3.44	−4.42	−4.56	−3.59	−1.46	
Numerical x	−0.163	−1.919	−3.422	−4.389	−4.521	−3.546	−1.453	+1.331

We see that no apology needs to be made for the accuracy of the graphical solution.

A word of explanation may be necessary in regard to scaling the values of x from the construction in Fig. 41. Take, for example, x_4 corresponding to $t = 4\Delta$. Dropping a perpendicular from the origin O to the radius P_4O_4, we obtain a point 4′ as shown. The scaled length P_44' gives x_4; it is positive because 4′ falls on the concave side of a loop OP_6 that

corresponds to positive values of the force $Q = F(t)$ in Fig. 38. Now consider the point $9'$ representing the projection of point O on the radius P_9O_9. This also falls on the concave side of loop P_6P_{13}; but since this loop corresponds to negative values of the force $Q = F(t)$ in Fig. 38, the displacement $x_9 = P_99'$ is negative. For the same reason, $P_{11}11' = x_{11}$ is a negative displacement. In general, a projection on the concave side of a loop indicates a displacement of the same sign as the disturbing force. Thus $P_{15}15'$ being on the convex side of a positive loop represents a negative value for x_{15}.

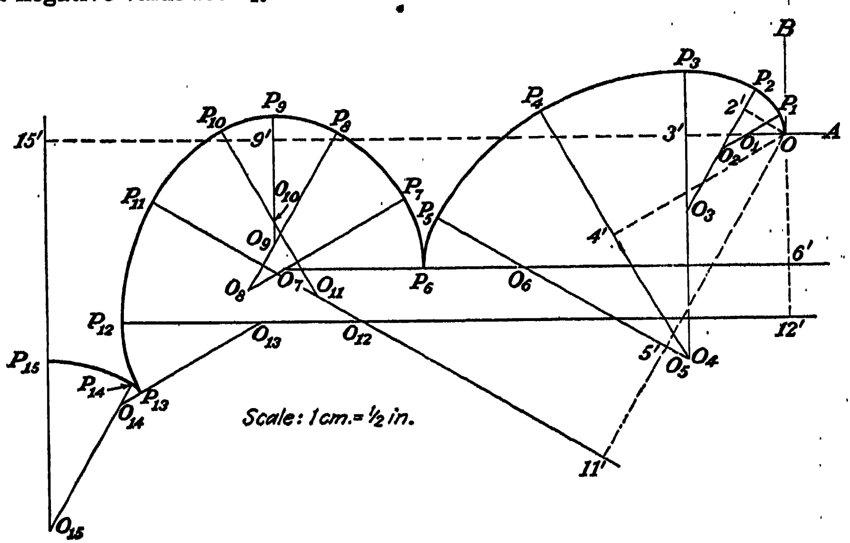


FIG. 41.

The graphical solution represented by Fig. 41 has several advantages over the numerical solution represented by Table IV: (1) It is much faster, and (2) it is independent of initial displacement x_0 and initial velocity $\dot{x}_0$. In our example, we assumed $x_0 = \dot{x}_0 = 0$. If we have some initial displacement and some initial velocity at $t = 0$, the numerical solution in Table IV must be made all over again, taking these initial conditions into account in the first horizontal line of the table. But in the graphical solution, as we see from Eq. (33d), x_0 and $\dot{x}_0/p$ are simply initial values of A and B. Hence to adapt the constructions in Fig. 41 to initial displacement x_0 and initial velocity $\dot{x}_0$, we have only to take a new origin O' such that the point O has coordinates $A_0 = x_0$ and $B_0 = \dot{x}_0/p$. Then projecting O' instead of O onto the various radii, we get values of x corrected for initial displacement and initial velocity.

Example: The graphical solution lends itself nicely to a discussion of the effect of Coulomb friction as a source of damping. Consider, for example, the system in

Fig. 42*a*, which has a natural period τ and an initial displacement x_0 as shown. During the first half cycle, while the weight W moves to the left, a constant friction force $F = \mu W$ opposes the motion as shown During the second half cycle, while motion is to the right, the force F reverses its direction and again opposes the motion. Thus, due to coulomb friction, we have a disturbing force $Q = F(t)$ as represented by the diagram in Fig. 42*b*. This is a step curve with $Q_i/k = \pm F/k$ and time interval $\Delta = \tau/2$; *i.e.*, $p\Delta = \pi$. Hence the graphic solution is represented by the system of alternately $\pm$ semicircular loops having the radii $\pm F/k$ as shown in Fig. 42*c*. The abscissas of points $P_1, P_2 \ldots$ represent the displacements at the ends of half cycles, and we see that the amplitude diminishes as an arithmetic progression, losing $2F/k$ in each half cycle. The motion ceases at the end of that half cycle for which P_n has the abscissa $|x_n| < |F/k|$, since in this position the spring force $-kx_n$ cannot overcome the friction force F. The method can also be used to discuss forced vibrations with Coulomb damping.[1]

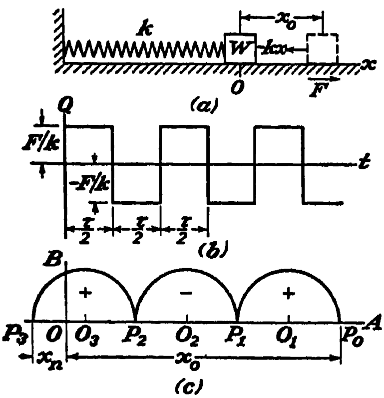


FIG. 42.

PROBLEMS

25. Develop the displacement-time equation for forced vibrations of an undamped system (Fig. 36*a*) under the action of a disturbing force $Q = Kt$. Assume $x_0 = \dot{x}_0 = 0$.

Ans. $x = (K/mp^3)\,[pt\,(1 - \cos pt) - \sin pt]$.

26. Verify expression (*h*), page 53, by using the graphical construction of Fig. 40.

27. Repeat the solution in Table IV for a system having the natural period $\tau = 0.8$ sec. Assume all other data to remain unchanged.

28. Solve Prob. 27 graphically.

29. For the system in Fig. 42*a*, $k = 10$ lb. per in., $W = 10$ lb., $\mu = 0.2$, and $x_0 = 2.0$ in., $\dot{x}_0 = 0$. How many half cycles of vibration will occur, and where will the block come to rest?

Ans. 10 half cycles; $x_{10} = 0$.

30. A wheel of weight W, pressed against an elastic foundation by a large constant force P, rolls with uniform horizontal velocity v as shown in Fig. 43. The spring constant for the foundation is k; and as long as the undisturbed foundation surface ahead of the wheel remains uniform and horizontal, the vertical deflection δ under the wheel remains constant and equal to $(P + W)/k$; *i.e.*, the wheel moves rectilinearly. Show that any slight irregularity in the undisturbed foundation surface, defined by the equation $\eta = f(x)$, will produce forced vibrations of the wheel on the foundation and that the differential equation of such vibrations will be

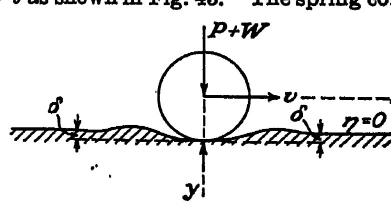


FIG. 43.

$$\frac{W}{g}\frac{d^2y}{dt^2} + ky = -\frac{W}{g}\frac{d^2\eta}{dt^2},$$

[1] See E. MEISSNER, "Resonanz bei konstanter Dämpfung," *Z. angew. Math. Mech.*, Band 15, Heft 1/2, February, 1935.

where y is the vertical displacement over and above that due to the static action of $P + W$. This example has found an important application in the study of vibrations of railroad car wheels due to slight irregularities in the track.[1]

8. Forced Nonlinear Vibrations.—In dealing with forced vibrations of a spring-suspended mass, we sometimes encounter systems where the restoring force is not a linear function of the displacement. For such cases, we consider the differential equation of motion (1) in the form

$$m\ddot{x} + E(x) + D(\dot{x}) = F(t), \tag{a}$$

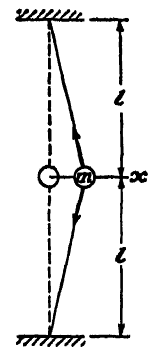


FIG. 44.

where $E(x)$ defines the restoring force, $D(\dot{x})$ the damping, and $F(t)$ the disturbing force. A discussion of this equation in its general form[2] is complicated; and for simplicity, we limit our discussion here to the symmetrical system shown in Fig. 44, the free vibrations of which, without damping, were already discussed in Art. 3. Using expression (f) on page 19, we have

$$E(x) = +\frac{2S_0}{l}x + \frac{AE}{l^3}x^3, \tag{b}$$

where S_0 is the initial tension in each wire; A its cross-sectional area; E its modulus of elasticity, and l its length. In addition, we now assume linear damping

$$D(\dot{x}) = c\dot{x} \tag{c}$$

and a simple harmonic disturbing force

$$F(t) = Q\cos(\omega t + \alpha). \tag{d}$$

Substituting expressions (b), (c), and (d) into Eq. (a) and dividing through by m, we obtain

$$\ddot{x} + \mu\dot{x} + p^2x + \beta x^3 = q\cos(\omega t + \alpha), \tag{37}$$

where

$$\mu = \frac{c}{m}, \qquad p^2 = \frac{2S_0}{lm}, \qquad \beta = \frac{AE}{l^3m}, \qquad q = \frac{Q}{m}. \tag{e}$$

Equation (37) is a nonlinear differential equation, the general solution of which is not known; in discussing it, we must proceed by approximate methods.

[1] See S. TIMOSHENKO, "Vibration Problems in Engineering," Van Nostrand, 2d ed., p. 107, 1937.

[2] For a more general method of dealing with this problem and a bibliography on the subject, see the paper by M. Raucher, Steady Oscillations of Systems with Non-linear and Unsymmetrical Elasticity, *J. Applied Mechanics*, pp. A-169 to A-177, December, 1938.

Confining our attention to a steady state of forced vibrations only and assuming that βx^3 is small compared with p^2x, we take, as an approximate solution[1] of Eq. (37),

$$x = a \cos \omega t. \tag{f}$$

This assumed solution represents the steady forced vibrations as a simple harmonic motion having the same angular frequency ω as the disturbing force. The relation between a and ω is left to be determined.

For this purpose, we substitute our trial solution (f), together with its derivatives, into Eq. (37) and obtain

$$-a\omega^2\cos \omega t - \mu a\omega \sin \omega t + p^2 a \cos \omega t + \beta a^3 \cos^3 \omega t$$
$$= q \cos \alpha \cos \omega t - q \sin \alpha \sin \omega t. \tag{g}$$

We see that this equation cannot be satisfied for all values of t; *i.e.*, expression (f) is not really a solution of Eq. (37). The best that we can do is to satisfy Eq. (g) for values of $\omega t = n\pi$, *i.e.*, for extreme positions of the vibrating mass; for intermediate positions, the equation will be only approximately satisfied. Taking $\omega t = n\pi$, for which $\cos \omega t = \pm 1$ and $\sin \omega t = 0$. Eq. (g) reduces to

$$p^2 a + \beta a^3 = q \cos \alpha + a\omega^2. \tag{38}$$

This algebraic equation defines an approximate relation between amplitude a, angular frequency ω, and phase angle α for steady forced vibrations of a given system.

To establish a second relationship between a, ω, and α, we must consider the question of the work done per cycle by the disturbing force and by the damping force. In order to have a steady state of vibration, the negative work of damping per cycle must be offset by an equal amount of positive work from the disturbing force. We consider first the work of the disturbing force.

$$W_F = \int_{x_t}^{x_{t+T}} F(t)dx = \int_0^{\frac{2\pi}{\omega}} F(t) \left(\frac{dx}{dt}\right) dt.$$

Substituting expression (d) for $F(t)$ and the first derivative of expression (f) for dx/dt and noting from notations (e) that $Q = qm$, we have

$$W_F = -mqa\omega \int_0^{\frac{2\pi}{\omega}} \cos(\omega t + \alpha) \sin \omega t \, dt = \pi mqa \sin \alpha. \tag{h}$$

From this result, we see that when the disturbing force leads the motion

[1] The justification for this lies in the fact that for $\beta = 0$, expression (f) can be an exact solution of Eq. (37) if the amplitude a is properly chosen. When $\beta \neq 0$ but is small, we can expect that the form of the motion will not differ greatly from expression (f).

$(0 < \alpha < \pi)$, it does positive work thereon and that this work is a maximum when $\alpha = \pi/2$, *i.e.*, when the force leads the motion by a quarter cycle. When the force lags behind the motion $(-\pi < \alpha < 0)$, it does negative work thereon. Thus in a steady state, we can always expect α to be positive and lie between 0 and π.

The energy dissipated by damping during one cycle will be

$$W_D = \int_{x_t}^{x_{t+\tau}} c\left(\frac{dx}{dt}\right)dx = \mu m \int_0^{\frac{2\pi}{\omega}} \left(\frac{dx}{dt}\right)^2 dt = \pi\mu m\omega a^2. \qquad (i)$$

We note that this dissipation of energy due to damping increases as the square of the amplitude while the energy input (h) increases only as the first power of the amplitude.

Now as stated above, the condition of a steady state of forced vibration requires that the negative work of damping shall be exactly offset by the positive work of the disturbing force. Equating (i) and (h), we find

$$\sin\alpha = \frac{\mu a\omega}{q}, \qquad (j)$$

from which

$$\cos\alpha = \pm\sqrt{1 - \frac{\mu^2}{q^2}(a\omega)^2}. \qquad (k)$$

This is the second required relation between the three variables a, ω, and α. Substituting expression (k) for $\cos\alpha$ in Eq. (38), we obtain

$$p^2a + \beta a^3 = \pm q\sqrt{1 - \frac{\mu^2}{q^2}(a\omega)^2} + a\omega^2. \qquad (39)$$

For a given system, *i.e.*, for given values of p^2, β, q, and μ, this algebraic equation defines the approximate relation between the amplitude a and the angular frequency ω for steady forced vibrations. It can best be solved by a graphical method that we shall now discuss.

For simplicity, we begin with a consideration of undamped vibrations, *i.e.*, we take $\mu = 0$, for which Eq. (39) reduces to

$$p^2a + \beta a^3 = \pm q + a\omega^2. \qquad (40)$$

Referring to expression (k) above, we see that the plus sign on q corresponds to $\alpha = 0$ while the minus sign corresponds to $\alpha = \pi$. Thus, without damping, the forced vibrations are either in phase or completely out of phase with the disturbing force. In either case, the work of the disturbing force per cycle is zero.

The left-hand side of Eq. (40) represents the restoring force per unit mass for an extreme position; the right-hand side is the sum of the dis-

turbing force and the inertia force, each per unit mass, for the same position. To satisfy Eq. (40), the amplitude a must be so chosen as to make these two quantities equal. This may be accomplished graphically as follows: First, from Eq. (b), we plot the spring force per unit mass as a function of amplitude a. This gives the curve OA in Fig. 45a. In

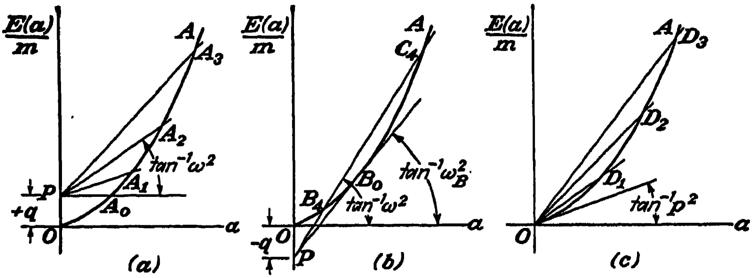


FIG. 45.

a similar manner, the right-hand side of Eq. (40) will be represented by straight lines having the intercept $\pm q$ on the vertical axis and slopes represented by various values of ω^2. The intersection between any such line and the curve OA determines, for the corresponding ω, the value (or values) of a for which Eq. (40) is satisfied. We consider first the case of in-phase vibrations ($\alpha = 0$) and take q with plus sign as shown. Then drawing several straight lines PA_1, PA_2, PA_3, . . . with the slopes ω_1^2, ω_2^2, ω_3^2, . . . , we get intersections A_1, A_2, A_3, . . . defining (by their abscissa) amplitudes a_1, a_2, a_3, ... Plotting a_i as ordinate against ω_i as abscissa, we get the curve A_0A_3 in Fig. 46 showing the amplitude of undamped in-phase forced vibrations for various frequencies $\omega/2\pi$ of the disturbing force.

Now let us consider the out-of-phase forced vibrations corresponding to a negative intercept q as shown in Fig. 45b. We see that in this case, there are no intersections for values of ω less than a certain value ω_B corresponding to the slope of the line PB_0 that is tangent to the curve OA at B_0, while for each line with slope greater than this, there are two intersections such as B_4 and C_4. Hence, for each value of $\omega > \omega_B$, there are two amplitudes of out-of-phase forced vibration that satisfy Eq. (40), Drawing several such lines and again plotting a_i against ω_i, we get the curve $C_4B_0B_5$ shown in Fig. 46.

Finally, if we draw lines OD_1, OD_2, OD_3, . . . through the origin O as shown in Fig. 45c, we get the solution of Eq. (40) for $q = 0$, *i.e.*, for free vibrations of the nonlinear system. The relation between a and ω obtained in this way is represented by the curve D_0D_5 in Fig. 46, which intersects the ω-axis at $\omega_0 = p$. This curve may be considered as an approximation to the reciprocal of the period-amplitude curve shown in

Fig. 18, page 23. The curves in Fig. 46 are called *response curves* for forced vibrations of the nonlinear system. In a general way, they correspond to the $\gamma = 0$ curves in Fig. 30, page 44, for linear forced vibrations without damping. Taking a series of values of q, a whole family of such curves can be drawn showing how the nonlinear system responds to various intensities and frequencies of the disturbing force.

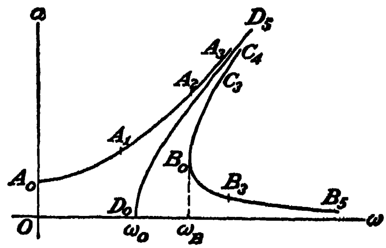


FIG. 46.

In the case of forced vibrations of a linear system as discussed in Art. 6, there was only one value of the amplitude for a given value of ω; in the case of a nonlinear system, the phenomenon is more complex. For angular frequencies of the disturbing force less than ω_B, there is again only one amplitude of in-phase forced vibration; but for frequencies larger than ω_B, there are three amplitudes, one corresponding to in-phase vibrations and the other two to out-of-phase vibrations. Experiments[1] show that of these two out-of-phase amplitudes, the larger one corresponds to an unstable motion that never occurs in an actual physical system.

For further discussion, we now return to the more general Eq. (39) which takes account of damping. The left-hand side of this equation is the same as that of Eq. (40) and is still represented by the curve OA in Fig. 45a. On the right, however, we now have a family of straight lines with the slopes ω^2 and variable intercepts on the vertical axis, as represented by the term containing the radical. Introducing the notation

$$q' = q\sqrt{1 - \frac{\mu^2}{q^2}(a\omega)^2}, \tag{l}$$

we see that the numerical value of this intercept in the case of damping is always smaller than q. However, its value depends upon the product $a\omega$; and since we are trying to find the proper relation between these two quantities, we must proceed by trial and error. To show how this works, we refer to Fig. 47a, where the straight line P_0A determines, for a chosen value of ω, the amplitude a_0 of undamped vibrations. Using this value of a_0 together with the chosen value of ω in expression (l), we compute a first approximation q_1' to the intercept q' which, of course, is smaller than q. Drawing a new line P_11 with the intercept q_1' and the same slope

[1] The first experiments of this kind were made with electric circuits by O. Martienssen. See *Physik. Z.*, vol. II, p. 448, 1910. Later experiments with mechanical vibrations were made by G. Duffing, "Erzuringene Schwingungen bei verändlicher Eigenfrequenz," Braunschweig, p. 40, 1918.

ω^2 as before, we get a somewhat smaller value a_1 for the amplitude. Now using the new product $a_1\omega$ in expression (l), we compute a second approximation q_2' to the intercept q' and draw the straight line P_22 as shown. Since $a_1\omega < a_0\omega$, we see that $q_1' < q_2' < q$; hence the amplitude

$$a_1 < a_2 < a_0.$$

By the same reasoning, we conclude that a third approximation to the amplitude must lie between a_1 and a_2; thus the process is convergent, and

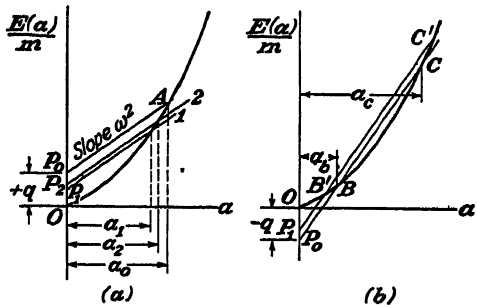


FIG. 47.

we soon find with sufficient accuracy the correct amplitude for the chosen value of ω.

Starting with $\omega = 0$ and proceeding in this way for a series of increasing values of ω, we can determine amplitudes in Fig. 47a and plot the curve A_0A_c shown in Fig. 48, representing the relation between a and ω for forced vibrations with a given amount of damping. Since a is increasing with ω along this branch of the response curve, it is evident that we must finally reach a point for which

$$\frac{\mu a\omega}{q} = 1, \tag{m}$$

and beyond this, expression (l) becomes imaginary. Thus, with damping, there is a limit to the amplitude of forced vibration of a nonlinear system. Using expressions (m) and (l), we see that this maximum amplitude is defined by the equations

$$a\omega = \frac{q}{\mu} \qquad \text{and} \qquad q' = 0. \tag{n}$$

Since the solution of Eq. (39) for $q' = 0$ corresponds to the free-vibration curve D_0D_5 in Fig. 48, the maximum amplitude and the corresponding angular frequency ω_c (point A_c) are determined by the intersection of this curve with the hyperbola $a\omega = q/\mu$. Finally, from expression (j), we see that this point corresponds to $\alpha = \pi/2$, and the disturbing force leads

the motion by a full quarter cycle and is doing its maximum positive work to offset damping. A steady state of forced vibration with larger amplitude cannot exist, and Eq. (39) gives no real solution for any larger amplitude.

To get the solution with damping corresponding to the minus sign in Eq. (39) and thereby complete the response curve in Fig. 48, we proceed in a similar manner. To illustrate, we refer to Fig. 47*b* and start with the intercept $-q$ and choose some value of $\omega > \omega_B$. The corresponding straight line through P_0 gives the amplitudes a_b and a_c for zero damping. Substituting either the product $a_c\omega$ or $a_b\omega$ in expression (*l*), we see that as before, the numerical value of q' will be less than q. Hence for a second approximation, we must draw a parallel line through some point P_1 nearer the origin. In this way, we get new intersections B' and C' defining new values of the amplitudes a_b' and a_c', as follows; $a_b' < a_b$ and $a_c' > a_c$. Using the product $a_c'\omega$ in expression (*l*), we get a somewhat smaller numerical value for q' and have to move the point P a little closer to the origin, which, in turn, gives a slightly larger amplitude a_c'', etc. The process, however, converges quite rapidly and we soon find the value of a_c that satisfies Eq. (39) for the chosen value of ω. Proceeding in this way, we can construct without much difficulty the portion A_cB_o of the response curve in Fig. 48. Since this portion of the curve represents an unstable state of motion, it has been indicated by a broken line in the figure.

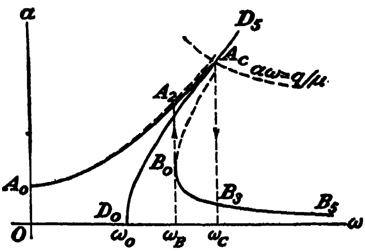


FIG. 48.

Regarding the B-intersections obtained in Fig. 47*b*, we see that, as in Fig. 47*a*, a reduction in the absolute value of q' also reduces the amplitude for a given ω; *i.e.*, $a_b' < a_b$ as noted above. Thus the correct B-intersection lies between B and B' and may be located in the same manner as in Fig. 47*a*. Proceeding in this way, we construct the curve B_0B_2 as shown in Fig. 48, and the response diagram is completed.

Referring to Fig. 48, we may now describe what may be expected to happen in the case of damped nonlinear forced vibrations if, starting with $\omega = 0$, we gradually increase the frequency of the disturbing force. At first, both the amplitude a and the phase angle α will grow with ω until we reach the value ω_C. Here, as we have already observed, the disturbing force leads the motion by a quarter cycle, and no further increase in amplitude can occur. As soon as ω is increased slightly beyond ω_C, the motion changes form completely; the amplitude falls to B_3, and the force leads the new motion by something more than a quarter cycle. With

further increase in ω, the amplitude gradually falls along the curve B_3B_5 and the phase angle approaches π. Now if we reverse the procedure and begin gradually to decrease ω, the amplitude grows according to the curve B_5B_0 until we reach ω_B. Since there is only one solution of Eq. (39) for values of $\omega < \omega_B$, a further decrease in ω will be accompanied by a rather sudden jump in amplitude and a change in phase angle from something greater than $\pi/2$ to something less than $\pi/2$. With further decrease in ω, the amplitude gradually falls according to the curve A_2A_0. We see that the question of instability exists only in the region $\omega_B < \omega < \omega_C$.

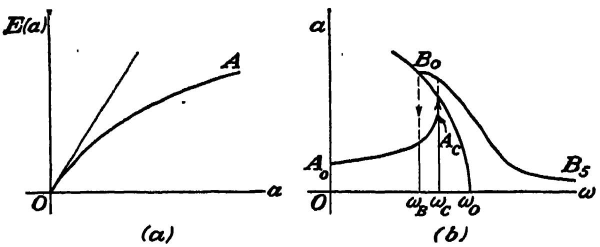


FIG. 49.

The foregoing discussion of nonlinear forced vibrations has been associated with the system in Fig. 44. It applies equally well, however, to any system in which the departure from linearity is not too great and which is symmetrical with respect to the middle position. That is, the curve OA in Fig. 45 or 47 need not be exactly the cubic parabola represented by Eq. (b). In general, if it has the same shape for displacements both sides of the middle position and the nonlinearity is not too great, we may proceed in the same manner. The method is, of course, always an approximate one, except in the case of a linear system. As a matter of fact, we can apply the method to a linear system and get the response curves shown in Fig. 30. In all cases, it is assumed that damping is not so great as to govern the motion (see page 35).

For the system in Fig. 44, the curve OA (Fig. 45) is concave upward; *i.e.*, the system becomes stiffer with increasing amplitude, and the natural frequency of free vibration rises. Such a system is said to be *hard*. Sometimes we have a so-called *soft system* for which the restoring force decreases with increasing amplitude. In such case, the curve OA is concave downward as shown in Fig. 49a. Proceeding as already explained, we get, for a soft system, response curves of the type shown in Fig. 49b.

In Fig. 50a, we have a symmetrical system for which the relation between restoring force and amplitude is discontinuous as shown in Fig. 50b. As long as the amplitude of vibration is less than a certain value δ, only the longer springs are in action and we have a linear system.

When the amplitude becomes larger than δ, the two middle springs come into action and the stiffness increases abruptly. If the change in slope

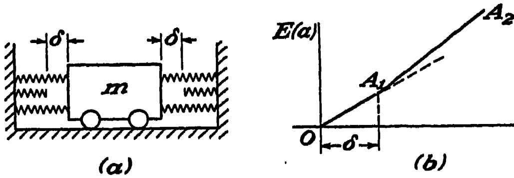


FIG. 50.

between OA_1 and A_1A_2 is not too great, the described method can be used also in this case with good accuracy.[1]

Finally in Fig. 51*a*, we have an unsymmetrical nonlinear system. For displacements of the mass m to the right of the equilibrium position, the spring characteristic is represented by the broken line OA_1A_2 in Fig. 51*b*; while for displacements to the left, we have the straight line OA'. In this case, the amplitudes of forced vibration on the hard and soft sides of the equilibrium position will be different, and the above method is not applicable without some modification. It is to such cases as this that the method developed by Raucher[2] is especially well suited.[3]

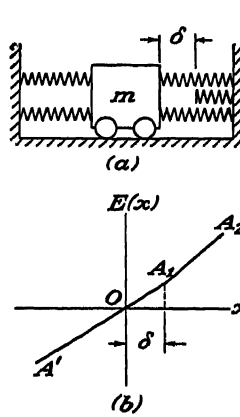


FIG. 51.

PROBLEMS

31. For the numerical data given in Table II, page 20, construct the curve OA and determine graphically, as in Fig. 45*c*, the approximate relation between amplitude a and natural frequency of free vibration of the system in Fig. 44. Compare the results with those given in Fig. 18, page 23.

32. Referring to Eq. (*b*) and Fig. 27, page 39, and assuming $mg = 1$ lb., $k = 1$ lb. per in., $c = 0.025$ lb. per in. per sec., and $Q_0 = 0.1$ lb., construct by graphical solution the response curve like those in Fig. 30, page 44.

33. Referring to the system in Fig. 44 and using the numerical data in Table II, page 20, construct to scale the response curves like those shown in Fig. 46. Assume $Q = 2$ lb.

9. Numerical Integration: Störmer's Method.—We have already noted in Art. 1 that the differential equation of rectilinear motion of a

[1] See JACOBSEN and JESPERSEN, Steady Forced Vibrations of Single Mass Systems with Symmetrical as Well as Unsymmetrical Non-linear Restoring Elements, *J. Franklin Inst.*, vol. 220, No. 4, 1935.

[2] RAUCHER, *op. cit.*, p. 61.

[3] For another method of dealing with this problem, see Étude graphique des vibrations de systèmes a un seul degré de liberté, by J. Lamoen, *Revue Universelle des Mines*, Vol. XI, No. 7, May, 1935.

particle may be nonlinear or contain coefficients varying with time. For example, we may have to deal with a vibrating mass for which the restoring force is not a linear function of displacement (Fig. 52a) or varies in some prescribed manner with time (Fig. 52b). In such cases, where damping is of secondary importance and can be neglected, the equation of motion takes the general form

$$\ddot{x} = \frac{1}{m} F(x,t), \qquad (a)$$

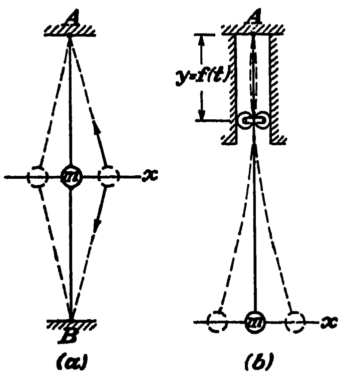


Fig. 52.

where $F(x,t)$ defines the resultant force and m is the mass of the particle. It is only for the simplest cases that we are able to obtain a rigorous solution of this equation; and in many practical problems, recourse must be made to some approximate method of integration. We take this opportunity to describe and show the application of one of the general *step-by-step methods* of numerical integration. The particular procedure to be described was developed about 1890 by Carl Störmer in connection with a mathematical theory of the aurora borealis. It has also proved very useful in the treatment of such practical problems as the motion of projectiles and trains.[1]

To apply the method, we write Eq. (a) in the form

$$\ddot{x} = \phi(x,t) \qquad (41)$$

and assume that for a small interval of time from 0 to t, the solution can be represented by the Maclaurin's series

$$x = x_0 + \dot{x}_0 t + \ddot{x}_0 \frac{t^2}{2} + \dddot{x}_0 \frac{t^3}{6} + \overset{\mathrm{IV}}{x}_0 \frac{t^4}{24} + \cdots . \qquad (42)$$

In this series, x_0 and $\dot{x}_0$ are the initial displacement and the initial velocity of the particle; these initial conditions must be known. The initial acceleration $\ddot{x}_0$ is found by substituting $x = x_0$ together with $t = 0$ in Eq. (41). In like manner, the values of $\dddot{x}_0$, $\overset{\mathrm{IV}}{x}_0$, . . . are obtained by making successive differentiation of Eq. (41) and using $x = x_0$ and $t = 0$ therein. Substituting all these initial values into the series (42), we can calculate the displacement x for a given value of t. If t is small, the series converges rapidly and the first few terms will give the value of x with sufficient accuracy for practical applications.

[1] For a more complete discussion of various methods of numerical integration together with applications, see "Sur l'integration numerique approchée des équations différentielles," by M. A.-N. Kriloff, Imprimerie Nationale, Paris, 1927.

Assume now that several values of x have been calculated in this manner for $t = 0$, $t = h$, $t = 2h$, . . . $t = nh$, where h is a small constant interval of time. These values of x, denoted by x_0, x_1, . . . x_{n-2}, x_{n-1}, x_n, are called *starting values;* they are arranged in tabular form as shown in column (2) of Table V. This done, we compute and record the first differences $\Delta x_{n-1} = x_n - x_{n-1}$, . . . as shown in column (3) and then the second differences $\Delta^2 x_{n-2} = \Delta x_{n-1} - \Delta x_{n-2}$, . . . as shown in column (4).

TABLE V

(1) Step	(2) x	(3) Δx	(4) $\Delta^2 x$	(5) ξ	(6) $\Delta\xi$	(7) $\Delta^2\xi$
0	x_0			ξ_0		
		. . .			. . .	
. . .	. . .		. . .	. . .		. . .
		. . .			. . .	
$n-2$	x_{n-2}		. . .	ξ_{n-2}		. . .
		Δx_{n-2}			$\Delta\xi_{n-2}$	
$n-1$	x_{n-1}		$\Delta^2 x_{n-2}$	ξ_{n-1}		$\Delta^2\xi_{n-2}$
		Δx_{n-1}			$\Delta\xi_{n-1}$	
n	x_n		(a) $\Delta^2 x_{n-1}$	ξ_n		(f) $\Delta^2\xi_{n-1}$
		(b) Δx_n			(e) $\Delta\xi_n$	
$n+1$	(c) x_{n+1}			(d) ξ_{n+1}		

All the above calculations are made for very small values of t, in which case the series (42) converges rapidly. For larger values of t, the series usually converges slowly and more and more terms must be taken to obtain a desired accuracy. Obviously, the amount of labor involved in such a procedure will very soon become prohibitive. The essential feature of Störmer's method is a means of prolonging the calculations with good accuracy even for large values of t without the necessity of using many terms of the series (42). For this purpose,

he introduces a quantity ξ, defined by the equation

$$\xi = h^2\ddot{x} = h^2\phi(x,t). \tag{43}$$

Substituting the starting values $x_0, x_1, \ldots, x_{n-2}, x_{n-1}, x_n$, together with $t = 0$, $t = h$, $\ldots$ $t = (n-2)h$, $t = (n-1)h$, $t = nh$, into this expression, we compute and record $\xi_0, \xi_1, \ldots \xi_{n-2}, \xi_{n-1}, \xi_n$, as shown in column (5) of Table V. Then, as before, we compute and record first and second differences $\Delta\xi$ and $\Delta^2\xi$ as shown in columns (6) and (7).

Having the quantities shown in boldface type in Table V, we are now ready to consider the prolongation of the calculations. To find the next value x_{n+1} of the displacement in column (2), we need the next second difference $\Delta^2 x_{n-1}$ in column (4). This is computed, as will be shown later, from the formula

$$\Delta^2 x_{n-1} \approx \xi_n + \tfrac{1}{12}(\Delta^2\xi_{n-2}). \tag{44}$$

Having this second difference [item (*a*) in column (4)], we easily obtain

$$\left.\begin{aligned} \Delta x_n &= \Delta x_{n-1} + \Delta^2 x_{n-1}, \\ x_{n+1} &= x_n + \Delta x_n, \end{aligned}\right\} \tag{b}$$

which are recorded in their proper places as shown [items (*b*) and (*c*)]. Substituting x_{n+1} together with $t = (n+1)h$ in expression (43), we now get ξ_{n+1} after which the differences $\Delta\xi_n$ and $\Delta^2\xi_{n-1}$ are computed and recorded in their proper places [items (*d*), (*e*), (*f*), in Table V]. After these six operations, made in the order (*a*), (*b*), (*c*), (*d*), (*e*), (*f*), we are ready to repeat the same procedure finding, in order, the quantities

$$\Delta^2 x_n,\ \Delta x_{n+1},\ x_{n+2}; \qquad \xi_{n+2},\ \Delta\xi_{n+1},\ \Delta^2\xi_n,$$

and so on.

We see that the success of the whole process, once the starting values have been obtained, rests on formula (44). Störmer derives this formula in the following manner. Considering a time $t = nh$ as a new origin and expanding $x = f(t)$ about this point, the series (42) may be written in the form

$$x = x_n + \dot{x}_n t_1 + \ddot{x}_n \frac{t_1^2}{2} + \dddot{x}_n \frac{t_1^3}{6} + \overset{\text{IV}}{x}_n \frac{t_1^4}{24} + \cdots, \tag{c}$$

where $t_1 = t - nh$. Substituting first $t_1 = +h$ and then $t_1 = -h$ in this expression, we obtain

$$\left.\begin{aligned} x_{n+1} &= x_n + \dot{x}_n h + \ddot{x}_n \frac{h^2}{2} + \dddot{x}_n \frac{h^3}{6} + \overset{\text{IV}}{x}_n \frac{h^4}{24} + \cdots, \\ x_{n-1} &= x_n - \dot{x}_n h + \ddot{x}_n \frac{h^2}{2} - \dddot{x}_n \frac{h^3}{6} + \overset{\text{IV}}{x}_n \frac{h^4}{24} - \cdots. \end{aligned}\right\} \tag{d}$$

Noting that $\ddot{x}_n h^2 = \xi_n$, we find from expressions (d) that

$$\Delta^2 x_{n-1} = x_{n+1} - 2x_n + x_{n-1} = \xi_n + \overset{\text{IV}}{x}_n \frac{h^4}{12} + \overset{\text{VI}}{x}_n \frac{h^6}{360} + \cdots,$$

or, limiting ourselves to the first two terms of this series,

$$\Delta^2 x_{n-1} \approx \xi_n + \overset{\text{IV}}{x}_n \frac{h^4}{12}. \qquad (e)$$

Now returning to the series (c) and differentiating twice with respect to time, we obtain

$$\ddot{x} = \ddot{x}_n + \dddot{x}_n t_1 + \overset{\text{IV}}{x}_n \frac{t_1^2}{2} + \cdots,$$

from which, by putting $t_1 = -h$ and $t_1 = -2h$, we find

$$\left.\begin{aligned} \xi_{n-1} &= h^2\ddot{x}_{n-1} = \xi_n - \dddot{x}_n h^3 + \overset{\text{IV}}{x}_n \frac{h^4}{2} - \cdots, \\ \xi_{n-2} &= h^2\ddot{x}_{n-2} = \xi_n - 2\dddot{x}_n h^3 + 4\overset{\text{IV}}{x}_n \frac{h^4}{2} - \cdots. \end{aligned}\right\} \qquad (f)$$

Again, limiting ourselves to the terms shown, we find

$$\Delta^2 \xi_{n-2} = \xi_n - 2\xi_{n-1} + \xi_{n-2} \approx 2\overset{\text{IV}}{x}_n \frac{h^4}{2}. \qquad (g)$$

Elimination of $\overset{\text{IV}}{x}_n$ between Eqs. (e) and (g) yields, finally,

$$\Delta^2 \xi_{n-2} \approx \xi_n + \tfrac{1}{12}(\Delta^2 \xi_{n-2}),$$

as already given by formula (44).

In the foregoing discussion, we have considered only second differences $\Delta^2\xi$ in the values of ξ. By using seven terms of the series (c), the method can be extended to work with fourth differences in ξ. In this case, formula (44) will be replaced by

$$\Delta^2 x_{n-1} = \xi_n + \tfrac{1}{12}(\Delta^2\xi_{n-2} + \Delta^3\xi_{n-3} + \tfrac{19}{20}\Delta^4\xi_{n-4}). \qquad (45)$$

From this expression, we see that the procedure outlined in Table V will be exact if third differences in ξ vanish and that if they do not, their magnitudes are an index to the error involved. By watching the variation in second differences $\Delta^2\xi$ as the work proceeds, one can judge the error involved and choose the interval h so as to attain the desired degree of accuracy.

In order to see what may be expected of the method in the way of accuracy, let us start with the equation of simple harmonic motion

$$\ddot{x} = -p^2 x \qquad (h)$$

and take as a particular case

$$p = 10\pi, \qquad x_0 = 1 \text{ in.}, \qquad \dot{x}_0 = 0.$$

In such case, the exact solution becomes

$$x = x_0 \cos pt \tag{i}$$

and the period $\tau = 2\pi/p = 0.2$ sec. In using Störmer's method, we shall divide this period into 20 steps and take $h = 0.01$ sec. With these data, formula (43) becomes

$$\xi = -0.01\pi^2 x = -0.0987x. \tag{j}$$

To save time, the starting values x_0, x_1, x_2 are computed from the known solution (i) in this case and are shown by boldface type in column (2) of Table VI below. The remainder of the calculations are made as explained above, and the table is completed up to $t = 0.1$ sec. $= \frac{1}{2}\tau$. For comparison, the exact values of x, computed from Eq. (i), are given in column (0). We see that even when we use only second differences in ξ and a comparatively large interval h, we have an error of less than $\frac{1}{2}$ per cent in the value of x at the end of a half cycle. Using fourth differences in ξ and formula (45), or cutting the interval h in half, a much greater accuracy can be obtained if desired.

TABLE VI

(0) $\cos pt$	(1) t	(2) x	(3) Δx	(4) $\Delta^2 x$	(5) ξ	(6) $\Delta\xi$	(7) $\Delta^2\xi$
1.0000	0	**1.0000**			**−0.0987**		
			−0.0489			**+ 48**	
0.9511	0.01	**0.9511**		**−0.0932**	**−0.0939**		**+93**
			−0.1421			**+141**	
0.8090	0.02	**0.8090**		−0.0790	**−0.0798**		+77
			−0.2211			+218	
0.5878	0.03	0.5879		−0.0574	−0.0580		+57
			−0.2785			+275	
0.3090	0.04	0.3094		−0.0300	−0.0305		+29
			−0.3085			+304	
0	0.05	0.0009		+0.0001	−0.0001		+ 1
			−0.3084			+305	
−0.3090	0.06	−0.3075		+0.0304	+0.0304		−29
			−0.2780			+274	
−0.5878	0.07	−0.5855		+0.0576	+0.0578		−57
			−0.2204			+217	
−0.8090	0.08	−0.8059		+0.0790	+0.0795		−77
			−0.1414			+140	
−0.9511	0.09	−0.9473		+0.0929	+0.0935		
			−0.0485				
−1.0000	0.10	−0.9958					

The method can be used also to calculate the period of free vibration of a nonlinear system like that shown in Fig. 52*a* and already discussed in Art. 3. In such cases, it is necessary only to carry the calculations to the point where x changes sign and then compute the quarter period by interpolation. Proceeding in this way, the period of vibration for a given amplitude can be found with very good accuracy.

As a second example, let us consider the system shown in Fig. 52*b* and assume that the movable support is so manipulated that the equation of motion for the attached mass becomes

$$\ddot{x} = -tx. \tag{k}$$

We also assume, as initial conditions, that

$$x_0 = 1 \text{ in.}, \qquad \dot{x}_0 = 0, \qquad \text{when } t = 0. \tag{l}$$

For small values of t, this motion will have a rather long period, and we choose, correspondingly, a much larger value of h than in the preceding example. Taking $h = 0.2$ sec., we have

$$\xi = h^2\ddot{x} = -0.04tx. \tag{m}$$

To obtain starting values, in this case, we first make several successive differentiations of Eq. (k) obtaining, in order,

$$\left.\begin{aligned} \dddot{x} &= -t\dot{x} - x, \\ \overset{\text{iv}}{x} &= -t\ddot{x} - 2\dot{x}, \\ \overset{\text{v}}{x} &= -t\dddot{x} - 3\ddot{x}, \\ \overset{\text{vi}}{x} &= -t\overset{\text{iv}}{x} - 4\dddot{x}, \\ &\cdots\cdots\cdots\cdots \end{aligned}\right\} \tag{n}$$

Then substituting the initial conditions (l) into expressions (k) and (n) and using the series (42), we obtain

$$x = 1 - \frac{t^3}{6} + \frac{t^6}{180} - \frac{t^9}{12{,}960} + \cdots. \tag{o}$$

From this series, the starting values of x, as shown by boldface type in column (2) of Table VII, have been computed. The corresponding values of ξ, computed from Eq. (m), are recorded in column (5). The remaining steps for the first 5 sec. of the motion are made in the prescribed manner, except that in this example, we have carried fourth differences in ξ and used formula (45) instead of (44). This requires but little additional work, and the improved accuracy is well worth the effort. The displacement-time curve plotted from the data in columns (1) and (2) in Table VII is shown in Fig. 53. We see that both the amplitude and the period of vibration are decreasing with time.

As we near the end of Table VII, we see that the corrections to ξ are beginning to affect the third figure in values of $\Delta^2 x$ in column (4). This is an indication that the interval h should be reduced if the calculations are to be prolonged much beyond $t = 4$ sec.

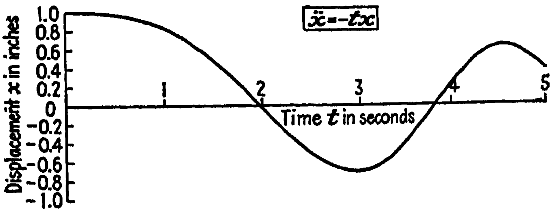


FIG. 53.

As a final example here, we refer again to Fig. 52*b* and assume that the movable support oscillates in such a way that the equation of motion for lateral vibrations of the mass m becomes[1]

$$\ddot{x} + (p^2 + \delta \sin \omega t)x = 0. \tag{46}$$

In this differential equation, $\delta \sin \omega t$ represents a *ripple* of period $\tau = 2\pi/\omega$ and amplitude δ that is superimposed upon the average value p^2 of the spring characteristic per unit of vibrating mass m. The nature of the motion represented by Eq. (46) will depend very much upon the ratio ω/p in any particular case, but the general solution is not known specifically in terms of p, δ, ω, x_0, and $\dot{x}_0$. To illustrate the numerical integration, we take as a particular case

$$2p = \omega = 2\pi, \qquad \tau = \frac{2\pi}{\omega} = 1 \text{ sec.}, \qquad \delta = \frac{p^2}{10}$$

and assume, as initial conditions,

$$x_0 = 1 \text{ in.}, \qquad \dot{x}_0 = 0, \qquad \text{when } t = 0.$$

Equation (46) may then be written

$$\ddot{x} = -p^2 x(1 + \tfrac{1}{10} \sin 2\pi t). \tag{p}$$

Making successive derivatives of this expression with respect to time and substituting the assumed initial conditions, we find

$$x_0 = 1, \qquad \dot{x}_0 = 0,$$

$$\ddot{x}_0 = -p^2, \qquad \dddot{x}_0 = -\frac{p^2}{5}, \qquad \overset{\text{IV}}{x}_0 = +p^4,$$

$$\overset{\text{V}}{x}_0 = \frac{8}{5}p^5, \qquad \overset{\text{VI}}{x}_0 = -\frac{21}{25}p^6, \quad \cdots,$$

[1] This differential equation is of considerable importance in various problems. It is called Mathieu's equation; see E. Mathieu, "Cours de physique mathematique," p. 122, Paris, 1873.

TABLE VII

(1) t	(2) x	(3) Δx	(4) $\Delta^2 x$	(5) ξ	(6) $\Delta\xi$	(7) $\Delta^2\xi$	(8) $\Delta^3\xi$	(9) $\Delta^4\xi$	(10) Σ
0	1.0000			0					
		−0.0013			−80				
0.2	0.9987		−0.0081	−0.0080		+2			
		−0.0094			−78		+3		
0.4	0.9893		−0.0156	−0.0158		+5		+3	+20
		−0.0250			−73		+6		
0.6	0.9643		−0.0232	−0.0231		+11		+2	+29
		−0.0482			−62		+8		
0.8	0.9161		−0.0291	−0.0293		+19		+3	+44
		−0.0773			−43		+11		
1.0	0.8388		−0.0334	−0.0336		+30		−5	+37
		−0.1107			−13		+6		
1.2	0.7281		−0.0345	−0.0349		+36		+2	+54
		−0.1452			+23		+8		
1.4	0.5829		−0.0323	−0.0326		+44		−6	+42
		−0.1775			+67		+2		
1.6	0.4054		−0.0255	−0.0259		+46		−3	+41
		−0.2030			+113		−1		
1.8	0.2024		−0.0142	−0.0146		+45		−11	+10
		−0.2172			+158		−12		
2.0	−0.0148		+0.0015	+0.0012		+33		−6	+3
		−0.2157			+191		−18		
2.2	−0.2305		+0.0204	+0.0203		+15		−9	−48
		−0.1953			+206		−27		
2.4	−0.4258		+0.0409	+0.0409		−12		−2	−72
		−0.1544			+194		−29		
2.6	−0.5802		+0.0599	+0.0603		−41		−6	−117
		−0.0945			+153		−35		
2.8	−0.6747		+0.0750	+0.0756		−76		+9	−119
		−0.0195			+77		−26		
3.0	−0.6942		+0.0823	+0.0833		−102		+10	−124
		+0.0628			−25		−16		
3.2	−0.6314		+0.0798	+0.0808		−118		+16	−103
		+0.1426			−143		0		
3.4	−0.4888		+0.0655	+0.0665		−118		+25	−44
		+0.2081			−261		+25		
3.6	−0.2807		+0.0396	+0.0404		−93		+21	+19
		+0.2477			−354		+46		
3.8	−0.0330		+0.0046	+0.0050		−47		+19	+101
		+0.2523			−401		+65		
4.0	+0.2193		−0.0349	−0.0351		+18		+11	+180
		+0.2174			−383		+76		
4.2	+0.4367		−0.0726	−0.0734		+94		−9	+219
		+0.1448			−289		+67		
4.4	+0.5815		−0.1008	−0.1023		+161		−17	+245
		+0.0440			−128		+50		
4.6	+0.6255		−0.1133	−0.1151		+211			
		−0.0693			+83				
4.8	+0.5562		−0.1048	−0.1068					
		−0.1741							
5.0	+0.3821								

TABLE VIII

(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)
t	x	Δx	$\Delta^2 x$	ξ	$\Delta\xi$	$\Delta^2\xi$	$\Delta^3\xi$	$\Delta^4\xi$	Σ
0	1.00000			−0.02467					
		−0.01244			− 45				
0.05	0.98756		−0.02505	−0.02512		+75			
		−0.03749			+ 30		+ 9		
0.1	0.95007		−0.02475	−0.02482		+84		− 6	+ 84
		−0.06224			+114		+ 3		
0.15	0.88783		−0.02352	−0.02368		+87		−10	+ 63
		−0.08576			+201		− 7		
0.2	0.80207		−0.02160	−0.02167		+80		− 4	+ 54
		−0.10736			+281		−11		
0.25	0.69471		−0.01881	−0.01886		+69		−11	+ 14
		−0.12617			+350		−22		
0.3	0.56854		−0.01532	−0.01536		+47		+ 1	+ 6
		−0.14149			+397		−21		
0.35	0.42705		−0.01138	−0.01139		+26		− 2	− 72
		−0.15287			+423		−23		
0.4	0.27418		−0.00716	−0.00716		+ 3		+ 4	− 31
		−0.16003			+426		−10		
0.45	0.11415		−0.00292	−0.00290		−16		+ 9	− 27
		−0.16295			+410		−10		
0.5	−0.04880		+0.00117	+0.00120		−26		+ 1	− 43
		−0.16178			+384		− 9		
0.55	−0.21058		+0.00502	+0.00504		−35		+11	− 20
		−0.15676			+349		+ 2		
0.6	−0.36734		+0.00849	+0.00853		−33		+ 2	− 23
		−0.14827			+316		+ 4		
0.65	−0.51561		+0.01167	+0.01169		−29		+ 1	− 18
		−0.13660			+287		+ 5		
0.7	−0.65221		+0.01454	+0.01456		−24		0	− 14
		−0.12206			+263		+ 5		
0.75	−0.77427		+0.01718	+0.01719		−19		− 6	− 27
		−0.10488			+244		− 1		
0.8	−0.87915		+0.01962	+0.01963		−20		− 3	− 31
		−0.08526			+224		− 4		
0.85	−0.96441		+0.02185	+0.02187		−24		− 7	− 53
		−0.06341			+200		−11		
0.9	−1.02782		+0.02384	+0.02387		−35		− 3	− 66
		−0.03957			+165		−14		
0.95	−1.06739		+0.02548	+0.02552		−40		− 2	− 83
		−0.01409			+116		−16		
1.0	−1.08148		+0.02663	+0.02668		−65		0	− 97
		+0.01254			+ 51		−16		
1.05	−1.06894		+0.02712	+0.02719		−81		+ 6	− 95
		+0.03966			− 30		−10		
1.1	−1.02928		+0.02681	+0.02689		−91		+ 7	− 90
		+0.06647			−121		− 3		
1.15	−0.96281		+0.02560	+0.02568		−94		+ 9	− 73
		+0.09207			−215		+ 6		
1.2	−0.87074		+0.02345	+0.02353		−88		+ 8	− 52
		+0.11552			−303		+14		
1.25	−0.75522		+0.02044	+0.02050		−74		+ 8	− 22
		+0.13596			−377		+22		
1.3	−0.61926		+0.01669	+0.01673		−52		+ 3	− 1
		+0.15265			−429		+25		

TABLE VIII.—(*Continued*)

(1) t	(2) x	(3) Δx	(4) $\Delta^2 x$	(5) ξ	(6) $\Delta\xi$	(7) $\Delta^2\xi$	(8) $\Delta^3\xi$	(9) $\Delta^4\xi$	(10) Σ
1.35	−0.46661		+0.01242	+0.01244		−27		− 3	+ 14
		+0.16507			−456		+22		
1.4	−0.30154		+0.00788	+0.00788		− 5		− 1	+ 36
		+0.17295			−461		+21		
1.45	−0.12859		+0.00328	+0.00327		+16		− 7	+ 37
		+0.17623			−445		+14		
1.5	+0.04764		−0.00115	−0.00118		+30		− 8	+ 34
		+0.17508			−415		+ 6		
1.55	+0.22272		−0.00530	−0.00533		+36		− 5	+ 33
		+0.16978			−379		+ 1		
1.6	+0.39250		−0.00909	−0.00912		+37		− 8	+ 15
		+0.16069			−342		− 7		
1.65	+0.55319		−0.01251	−0.01254		+30		+ 3	+ 25
		+0.14818			−312		− 4		
1.7	+0.70137		−0.01565	−0.01566		+26		0	+ 18
		+0.13253			−286		− 4		
1.75	+0.83390		−0.01850	−0.01852		+22		+ 2	+ 20
		+0.11403			−264		− 2		
1.8	+0.94793		−0.02114	−0.02116		+20		+ 8	+ 40
		+0.09289			−244		+ 6		
1.85	+1.04082		−0.02358	−0.02360		+26		+ 5	+ 53
		+0.06931			−218		+11		
1.9	+1.11013		−0.02575	−0.02578		+37		+ 6	+ 77
		+0.04356			−181		+17		
1.95	+1.15369		−0.02755	−0.02759		+54		− 1	+ 85
		+0.01601			−127		+16		
2.0	+1.16970		−0.02880	−0.02886		+70		+ 2	+108
		−0.01279			− 57		+18		
2.05	+1.15691		−0.02936	−0.02943		+88		− 8	+100
		−0.04215			+ 31		+10		
2.1	+1.11476		−0.02903	−0.02912		+98		− 6	+100
		−0.07118			+129		+ 4		
2.15	+1.04358		−0.02775	−0.02783		+102		−11	+ 77
		−0.09893			+231		− 7		
2.2	+0.94465		−0.02544	−0.02552		+95		− 8	+ 57
		−0.12437			+326		−15		
2.25	+0.82028		−0.02220	−0.02226		+80		− 8	+ 26
		−0.14657			+406		−23		
2.3	+0.67371		−0.01815	−0.01820		+57		− 4	− 1
		−0.16472			+463		−27		
2.35	+0.50899		−0.01355	−0.01357		+30		+ 2	− 18
		−0.17827			+493		−25		
2.4	+0.33072		−0.00864	−0.00864		+ 5			
		−0.18691			+408				
2.45	+0.14381		−0.00368	−0.00366					
		−0.19059							
2.5	−0.04678								

and the series (42) becomes

$$x = 1 - \frac{(pt)^2}{2} - \frac{(pt)^3}{30} + \frac{(pt)^4}{24} + \frac{(pt)^5}{75} - \frac{7(pt)^6}{6{,}000} - \cdots \qquad (q)$$

Taking $h = 0.05$ sec., five starting values of x are now computed from

this series and recorded as shown by boldface type in column (2) of Table VIII. For further calculation, we write

$$\xi = h^2\ddot{x} = -0.024674x(1 + \tfrac{1}{10}\sin 2\pi t). \tag{r}$$

The values of ξ corresponding to the starting values of x are shown by boldface type in column (5). Proceeding further on the basis of formula (45), the step-by-step calculations are made in the usual manner and are shown as far as $t = 2.5\tau = 2.5$ sec.

Having the numerical calculations in Table VIII, we shall now proceed to show how the displacement x for any time t can be found without the necessity of prolonging further the step-by-step computations. The possibility to do this rests on the fact that Eq. (46) has a general solution of the form[1]

$$x = C_1\phi(t)e^{\mu t} + C_2\psi(t)e^{-\mu t}, \tag{47}$$

in which C_1 and C_2 are constants of integration, μ is a quantity independent of t, and $\phi(t)$ and $\psi(t)$ are periodic functions of time, having the same period τ as the function $\delta \sin \omega t$ in Eq. (46). From expression (47), we see that if the displacement x for any instant t' is

$$x_0' = a' + b',$$

then, in general, for the instants $t', t' + \tau, t' + 2\tau, \ldots t' + n\tau$, we must have

$$\left.\begin{aligned}
x_0' &= a' + b',\\
x_1' &= sa' + \frac{1}{s}b'\\
x_2' &= s^2a' + \frac{1}{s^2}b',\\
&\ldots\ldots\ldots\ldots\ldots,\\
&\ldots\ldots\ldots\ldots\ldots,\\
x_n' &= s^na' + \frac{1}{s^n}b',
\end{aligned}\right\} \tag{48}$$

where

$$s = e^{\mu\tau} \tag{s}$$

is a constant independent of t'. From the first three of Eqs. (48) we easily find, corresponding to the chosen instant t',

$$a' = \frac{sx_1' - x_0'}{s^2 - 1}, \qquad b' = \frac{sx_1' - x_2'}{1 - \dfrac{1}{s^2}} \tag{49}$$

and also the quadratic equation

$$s^2 - \frac{x_0' + x_2'}{x_1'}s + 1 = 0, \tag{t}$$

from which

$$s = \frac{x_0' + x_2'}{2x_1'} \pm \sqrt{\left(\frac{x_0' + x_2'}{2x_1'}\right)^2 - 1}. \tag{50}$$

Substituting the values

$$x_0 = 1.00000, \qquad x_1 = -1.08148, \qquad x_2 = 1.16970 \tag{u}$$

[1] See FLOQUET, "Annales de l'École normale," vol. 1883/84.

from Table VIII into expression (50), we find, in our case,

$$s_1 = -1.0821 \qquad \text{or} \qquad s_2 = -0.9241. \tag{v}$$

Since one of these values is the reciprocal of the other, it makes no difference which we use. Changing s-values simply interchanges the quantities a' and b', and Eqs. (48) are unaffected. Confining our attention then to $s_1 = -1.082$, we find from Eqs. (49)

$$a_0 = 1.000, \qquad b_0 = 0.000.$$

Thus the displacement corresponding to $t = n\tau$ is

$$x_n = (-1.082)^n \text{ in.}$$

We see that the amplitude is growing with time, and the motion is said to be *unstable*.

We can expect from Eq. (s) that the value of s is independent of which three periodic values of x from Table VIII we use to compute it. To check our numerical integration, we now use in Eq. (50) the values

$$x_0' = -0.04880, \qquad x_1' = +0.04764, \qquad x_2' = -0.04678 \tag{w}$$

taken from Table VIII for $t = 0.5, 1.5, 2.5$ sec. With these values we get

$$s_1 = -1.0826 \qquad \text{or} \qquad s_2 = -0.9237,$$

which agrees very well with the values (v) above.

Although the value of s is constant and independent of t', the quantities a' and b' defined by Eqs. (49) are not, since they contain the periodic functions $\phi(t')$ and $\psi(t')$ and also $e^{\mu t'}$ and $e^{-\mu t'}$. For example, using the values (w) in Eqs. (49), we get

$$a_{0.5}' = -0.0161 \qquad \text{and} \qquad b_{0.5}' = -0.0327.$$

Hence, for any time $t = 0.5 + n\tau$, we have

$$x_n = -0.0161(-1.082)^n - 0.0327\left(\frac{1}{-1.082}\right)^n.$$

In a similar way, the value of x for any time $t = t' + n\tau$ can be found by taking the proper values for x_0', x_1', x_2' from Table VIII.

Sometimes, in dealing with the numerical integration of Eq. (41), it may happen that the function $\phi(x,t)$ is not given in analytic form and successive differentiations cannot be made. In such cases, we may take, at the beginning, a sufficiently small time interval h that we will be justified in using only three terms of the series (42) to compute starting values x_0, x_1, x_2. Then continuing with the step-by-step integration up to $t = 4h$, we get values x_0, x_1, x_2, x_3, x_4. After this, we double the interval and take x_0, x_2, x_4 as new starting values and proceed again up to $t = 4(2h)$. Then the interval can be doubled again, etc., until a suitable working interval is reached.

PROBLEMS

34. Repeat the calculations in Table VI, using fourth differences in ξ and formula (45), and show how much the accuracy will be improved.

35. Repeat the calculations in Table VI, using $h = 0.005$ sec. and only second differences in ξ. Compare the accuracy of these calculations with that of the preceding problem.

36. Using the data in Table VIII and formulas (48) and (49), calculate the displacement x for $t = 5.25$ sec. Use $s = -1.082$. *Ans.* $x_{5.25} = -1.051$ in.

10. Plane Harmonic Motion.—In discussing the curvilinear motion of a particle, we begin with Newton's second law of motion, expressed by the equation

$$m\bar{a} = \bar{F}, \tag{a}$$

where m is the mass of the particle, $\bar{F}$ is the resultant acting force, and $\bar{a}$ is the corresponding acceleration. Equation (a) is a vector equation and states that under the action of the force F, the particle receives an acceleration $\bar{a}$ that is in the same direction as the force and proportional to its magnitude. We recall that Newton's second law assumes no limitations regarding the motion of the particle before the force $\bar{F}$ begins to act. Thus, as concluded in Art. 1, when $\bar{F}$ is constant in direction and any *initial motion* of the particle is in the same direction, we obtain the particular case of rectilinear motion. If the direction of the resultant force varies, or if the particle has some initial motion in a direction not coinciding with that of the force, we obtain a *curvilinear motion*. Thus, for example, a projectile fired horizontally into the vertical force field of gravity describes a curvilinear path. In many practical problems, the motion will be confined to one plane and we speak of *plane curvilinear motion*. In such cases (Fig. 54), we resolve the force $\bar{F}$ and the acceleration $\bar{a}$ into rectangular components X, Y and $\ddot{x}$, $\ddot{y}$, respectively, and write

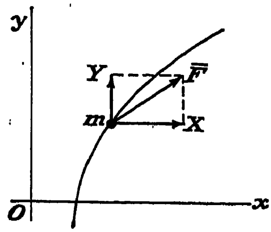


FIG. 54.

$$\left.\begin{aligned} m\ddot{x} &= X, \\ m\ddot{y} &= Y. \end{aligned}\right\} \tag{51}$$

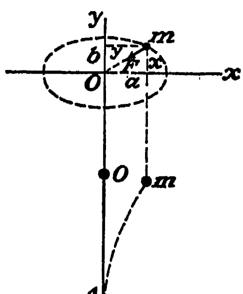


FIG. 55.

These are the differential equations of plane curvilinear motion of a particle. In the simplest cases, Eqs. (51) are independent equations and can be integrated separately. As an example, we shall discuss here the problem of free vibrations in a plane.

In Fig. 55, we have a heavy particle of mass m attached to the upper end of a slender vertical wire of uniform circular cross section that is built-in at A as shown. In discussing motion of the particle m in the horizontal plane,[1] we choose rectangular coordinate axes x and y having their origin O in coincidence with the equilibrium position of the particle.

[1] We assume small lateral displacements and neglect the corresponding insignificant vertical displacements; we also neglect air resistance and inertia of the wire.

If the particle be given a small initial displacement or initial velocity along the x-axis and released, it will perform simple harmonic motion in the x-direction. The same will be true for an initial displacement or velocity in the y-direction. However, if we give to the particle an initial displacement x_0 and release it with an initial velocity $\dot{y}_0$, it will describe a curvilinear path in the xy-plane. To study this motion, let us consider the particle in any position defined by the coordinates x and y (Fig. 55). Denoting by δ the lateral deflection of the end of the wire and by k the spring constant, the resultant force acting on the particle in this position is $\bar{F} = k\delta$ directed toward the origin O. The projections of this force are

$$X = -kx, \qquad Y = -ky,$$

and the equations of motion (51) become

$$\left.\begin{aligned} m\ddot{x} &= -kx, \\ m\ddot{y} &= -ky. \end{aligned}\right\} \tag{b}$$

Using the notation $p^2 = k/m$ and integrating Eqs. (b), we obtain

$$\left.\begin{aligned} x &= A \cos pt + B \sin pt, \\ y &= C \cos pt + D \sin pt, \end{aligned}\right\} \tag{c}$$

where A, B, C, D are constants of integration. Substituting the initial conditions

$$(x)_{t=0} = x_0, \qquad (\dot{x})_{t=0} = 0, \qquad (y)_{t=0} = 0, \qquad (\dot{y})_{t=0} = \dot{y}_0 \tag{d}$$

in Eqs. (c), we find

$$A = x_0 = a, \qquad B = 0, \qquad C = 0, \qquad D = \frac{\dot{y}_0}{p} = b,$$

and the equations of the path, in parametric form, become

$$\left.\begin{aligned} x &= a \cos pt, \\ y &= b \sin pt. \end{aligned}\right\} \tag{e}$$

Eliminating t between Eqs. (e), we find for the equation of the path of the particle in the xy-plane

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1. \tag{f}$$

This is an ellipse with major and minor semiaxes a and b, and the motion is called *elliptic harmonic motion;* the period $\tau = 2\pi/p$.

If the vertical wire to which the particle m is attached is of rectangular cross section so that it has two different flexural rigidities in two

principal planes, and if the principal axes of the cross section are taken as the x- and y-axes, the equations of motion become

$$m\ddot{x} = -kx, \qquad m\ddot{y} = -k_1 y, \tag{g}$$

where k and k_1 are two different spring constants corresponding to the two different flexural rigidities of the wire. Proceeding as before with the integration of these equations and using the same initial conditions (d), we obtain

$$x = a \cos pt, \qquad y = b \sin p_1 t, \tag{h}$$

where

$$p = \sqrt{\frac{k}{m}}, \qquad p_1 = \sqrt{\frac{k_1}{m}}, \qquad a = x_0, \qquad b = \frac{\dot{y}_0}{p_1}.$$

In this case, the equation of the path is more complicated and its shape depends upon the relation between the two angular frequencies p and p_1.

Let us assume first that $p_1 = 2p$. Then making this substitution in the second of Eqs. (h) and eliminating the parameter t as before, we obtain, for the equation of the path,

$$y = \pm 2 \frac{bx}{a} \sqrt{1 - \frac{x^2}{a^2}}. \tag{i}$$

This is the equation of the curve shown in Fig. 56a. Since the period $\tau_x = 2\pi/p$ is two times larger than the period $\tau_y = 2\pi/p_1$, there are two complete oscillations in the y-direction to one in the x-direction. Again, if $p_1 = 3p$, there are three complete oscillations in the y-direction to one in the x-direction and we obtain the path shown in Fig. 56b.

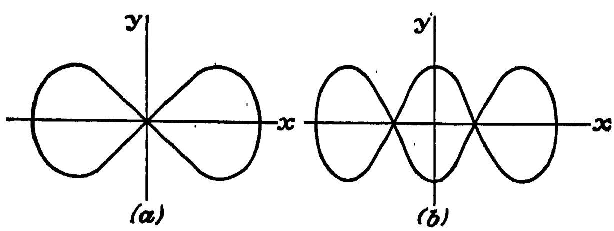


FIG. 56.

If the cross section of the wire is very nearly square so that the two spring constants k and k_1, corresponding to the two principal planes of bending, are almost, but not quite, equal, we obtain an interesting variation of the motion represented in Fig. 55. Assume, for example, that k_1 is slightly larger than k, so that $p_1 = p + \Delta p$. In this case, Eqs. (h) take the form

$$x = a \cos pt, \qquad y = b \sin(pt + \Delta pt). \tag{j}$$

Eliminating pt between these two expressions, we obtain the equation of

the path of the particle in the form

$$\frac{x^2}{a^2} - \frac{2xy}{ab}\sin \Delta pt + \frac{y^2}{b^2} = \cos^2\Delta pt. \tag{k}$$

This is the equation of an ellipse inscribed in the rectangle formed by the straight lines $x = \pm a$ and $y = \pm b$ (Fig. 57). The orientation of this ellipse depends upon the value of the angle Δpt; and since this angle increases uniformly with time, the shape and orientation of the ellipse within its circumscribed rectangle are continually changing.

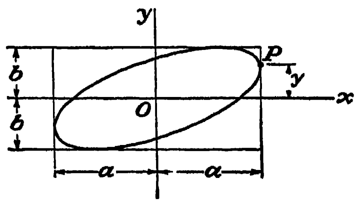


FIG. 57.

A picture of the cyclic changes through which the ellipse in Fig. 57 passes may be obtained as follows: Referring to Fig. 58a, let $ABB'A'$ be a transparent right circular cylinder of radius b and altitude $2a$, on the surface of which is traced the ellipse representing the diagonal plane section AB'. Now if this cylinder is rotated uniformly about its geometric axis OO at the rate of Δp rad. per sec., we shall see the ellipse traced on its surface go through a series of cyclic changes. First, let us say we are looking at the edge of the plane of the ellipse so that we see simply a straight line AB' or, more properly, the two straight lines AB' and $B'A$ superimposed on one another (Fig. 58a). Gradually these lines open up into a narrow ellipse AB' (Fig. 58b); and finally, after the cylinder has made a quarter turn, we shall see the fully opened ellipse in Fig. 58c. With further rotation of the cylinder, the ellipse gradually contracts (Fig. 58d) until after a half turn, we see again the straight line AB' across the other diagonal of the rectangle. This line then opens into the ellipse in Fig. 58c and finally closes up into the line AB' in Fig. 58a. A complete revolution of the cylinder has now been made, and the same series of changes begins over again. Obviously the period of the cycle is $\tau = 2\pi/\Delta p$.*

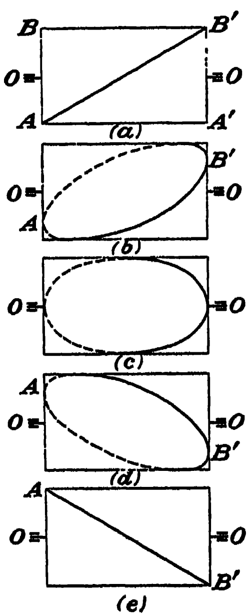


FIG. 58.

To show the connection between the curves represented by Eq. (k)

* The above-described transparent cylinder for demonstrating the cyclic changes in the plane curve represented by Eq. (k) is known as Lissajous' cylinder, and in general, such curves are called *Lissajous' figures*.

and the various aspects of the ellipse in Fig. 58, we return to Eqs. (j). Holding x constant and equal to a, we see from the first of these equations that $\cos pt = 1$, and hence the second equation becomes

$$y = b \sin \Delta pt.$$

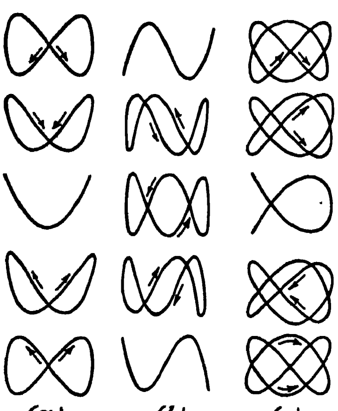
This shows that the point of tangency P in Fig. 57 moves along the line $x = a$ with simple harmonic motion of period $\tau = 2\pi/\Delta p$, which is obviously the motion of point B' in Fig. 58. The same argument can be made for any other point on the curve in Fig. 57 and the corresponding point on the ellipse of Lissajous' cylinder.

Similar cyclic variations of the curves in Fig. 56 occur when the frequency in the y-direction is very close to, but not quite, two or three times the frequency in the x-direction. If, for example, the rectangular cross section of the wire to which the particle is attached is of such proportions that $p_1 = 2p + \Delta p$, Eqs. (h) take the form

$$\left.\begin{aligned} x &= a \cos pt, \\ y &= b \sin(2pt + \Delta pt). \end{aligned}\right\} \tag{l}$$

Eliminating pt again, we obtain, for the equation of the path of the particle,

$$\frac{y}{b} = \frac{2x}{a}\sqrt{1 - \frac{x^2}{a^2}} \cos \Delta pt - \left(1 - \frac{2x^2}{a^2}\right) \sin \Delta pt. \tag{m}$$



(a) (b) (c)

FIG. 59.

When $\Delta pt = 0$ or $\Delta pt = \pi$, this equation reduces to

$$y = \pm \frac{2b}{a} x \sqrt{1 - \frac{x^2}{a^2}},$$

which is the same as Eq. (i) above. When $\Delta pt = \pi/2$ or $\Delta pt = 3\pi/2$, Eq. (m) reduces to

$$y = \pm b\left(1 - \frac{2x^2}{a^2}\right),$$

which represents two parabolas symmetrical with respect to the y-axis. The cyclic changes through which the path passes during the half period $\pi/\Delta p$ are represented in Fig. 59a. Two other cases, $p_1 = 3p + \Delta p$ and $2p_1 = 3p + \Delta p$, are shown in Fig. 59b and 59c, respectively. The arrows show the direction of motion of the particle along the path. All these figures can be demonstrated by inscribing

the proper curve on the surface of a transparent cylinder and then rotating it slowly about its axis.

Another simple method for demonstrating the cyclic variations of the curves in Fig. 59 may be had by attaching a heavy steel ball painted with luminous paint to the end of a thin flat steel strip twisted through 90 deg. in the middle as shown in Fig. 60*a* and then fixing the lower end in a vise. In this way, the frequency of vibration of the ball in the direction normal to the plane of the figure will depend practically only on the length l_1, while the frequency of vibration in the plane of the figure will depend practically only on the length l_2. Thus by adjusting the length l_2, the frequencies corresponding to the two principal planes of bending of the strip can be made very nearly equal or one very nearly equal to twice the other, etc. When vibrations of the luminous ball are observed in a dark room, the outline of the path can be clearly seen.

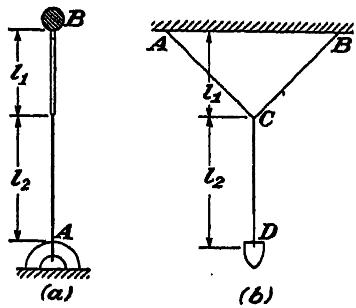


FIG. 60.

Figure 60*b* represents another device for tracing Lissajous' figures. Here we have a small cup filled with fine sand and suspended by strings as shown. When the cup is swinging in the plane of the paper, the period corresponds to the length l_2, while normal to the plane of the paper, the period corresponds to the length $l_1 + l_2$. If this pendulum is given both motions simultaneously, it will describe some such path as those we have been discussing. If the sand can trickle out slowly through a small hole in the bottom of the cup, a trace of the path can be obtained.[1]

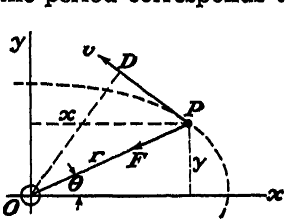


FIG. 61.

11. Planetary Motion.—In discussing the motion of a planet P around the sun O (Fig. 61), we shall neglect, in comparison with the attraction of the sun, the attraction of other planets and shall assume further that the sun represents a fixed point in space. Thus we have motion of a particle under the action of a so-called *central force* F that is always directed toward the sun, which we choose as the origin of our coordinate system.

Since the law of gravitation states that the attraction between two bodies is proportional to the product of their masses and inversely pro-

[1] For some pictures of various Lissajous' figures, see "The Scientist as Artist" by Robert T. Lagemann, American Scientist, Vol. 36, No. 3, July, 1948.

portional to the square of the distance between them, we may write

$$F = \frac{\mu}{r^2} m, \tag{a}$$

where $\mu = kM$ is a constant times the mass M of the sun, r is the distance between sun and planet, and m is the mass of the planet. The projections of this resultant force on the coordinate axes x and y are

$$X = -\frac{\mu m}{r^2}\frac{x}{r} \quad \text{and} \quad Y = -\frac{\mu m}{r^2}\frac{y}{r}; \tag{b}$$

and since $r = \sqrt{x^2 + y^2}$, the equations of motion (51) become

$$\left.\begin{aligned} m\ddot{x} &= -\frac{\mu m x}{(x^2 + y^2)^{\frac{3}{2}}}, \\ m\ddot{y} &= -\frac{\mu m y}{(x^2 + y^2)^{\frac{3}{2}}}. \end{aligned}\right\} \tag{c}$$

These equations are more complicated to integrate than those of the preceding article because each of them contains both x and y and they cannot be handled separately.

Multiplying the first equation by y and the second by x and subtracting the first from the second, we obtain

$$m(\ddot{y}x - \ddot{x}y) = 0 \tag{d}$$

This expression is equivalent to

$$\frac{d}{dt}(\dot{y}x - \dot{x}y) = 0,$$

from which we deduce the important relationship[1]

$$\dot{y}x - \dot{x}y = \text{const.} = h. \tag{e}$$

With the aid of this expression, Eqs. (c) can be integrated without much difficulty.

We begin by writing them in the following form:

$$\ddot{x} = -\frac{\mu x}{r^2 r}; \qquad \ddot{y} = -\frac{\mu y}{r^2 r}. \tag{f}$$

Then denoting by θ the angle that the radius vector OP makes with the x-axis at any instant, we introduce the following change in variables:

$$x = r\cos\theta; \qquad y = r\sin\theta. \tag{g}$$

[1] Expression (e) represents a particular case of the principle of angular momentum, which is discussed fully in Art. 15, p. 119. See also the authors' "Engineering Mechanics," 2d ed., p. 366.

Using notations (g), Eq. (e) becomes

$$r^2\dot{\theta} = h, \tag{h}$$

where $\dot{\theta} = d\theta/dt$. Substituting $\dot{\theta}/h$ for $1/r^2$ in Eqs. (f), they become

$$\left.\begin{aligned} \ddot{x} &= -\frac{\mu}{h}\frac{x}{r}\dot{\theta} = -\frac{\mu}{h}\cos\theta\,\dot{\theta}, \\ \ddot{y} &= -\frac{\mu}{h}\frac{y}{r}\dot{\theta} = -\frac{\mu}{h}\sin\theta\,\dot{\theta}, \end{aligned}\right\} \tag{i}$$

which, in turn, may be written as follows:

$$\left.\begin{aligned} \ddot{x} &= -\frac{\mu}{h}\frac{d}{dt}\sin\theta = -\frac{\mu}{h}\frac{d}{dt}\left(\frac{y}{r}\right), \\ \ddot{y} &= +\frac{\mu}{h}\frac{d}{dt}\cos\theta = +\frac{\mu}{h}\frac{d}{dt}\left(\frac{x}{r}\right). \end{aligned}\right\} \tag{j}$$

Integrating Eqs. (j), we obtain

$$\dot{x} = -\frac{\mu y}{hr} + c, \qquad \dot{y} = +\frac{\mu x}{hr} + d, \tag{k}$$

where c and d are constants of integration.

If, for each position of the planet P along its path in Fig. 61, we lay out from a fixed origin O (Fig. 62) the corresponding velocity vector $\bar{v}$, the locus of the ends of these vectors defines a curve AB, called the *hodograph*[1] of the planet. Since the rectangular coordinates of any point on this curve are the components $\dot{x}$ and $\dot{y}$ of the velocity, we see that Eqs. (k) define this hodograph. Replacing y and x in these equations by the values (g) and then squaring and adding together, we obtain

$$(\dot{x} - c)^2 + (\dot{y} - d)^2 = \frac{\mu^2}{h^2}. \tag{l}$$

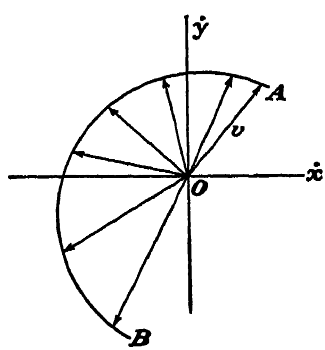


FIG. 62.

From Eq. (l), we see that the hodograph is a circle of radius μ/h and center coordinates c and d in the $\dot{x}\dot{y}$-plane.

The equation of the path of the planet can now be obtained from Eqs. (k) defining the hodograph. Multiplying the first by $y = r\sin\theta$ and the second by $x = r\cos\theta$ and subtracting the first from the second, we obtain, with the help of Eq. (e),

$$\dot{y}x - \dot{x}y = r\left(\frac{\mu}{h} + d\cos\theta - c\sin\theta\right) = h. \tag{m}$$

[1] See *ibid.*, p. 345.

This equation gives the relation between the radius vector r and the angle θ. Thus it represents, in polar coordinates, the desired equation of the path of the planet. To simplify the expression, we introduce for the coordinates c and d of the center of the circular hodograph the notations

$$-c = b \sin \alpha, \qquad d = b \cos \alpha, \tag{n}$$

where b and α are simply new constants. Then Eq. (m) becomes

$$r\left[\frac{\mu}{h} + b \cos(\theta - \alpha)\right] = h, \tag{o}$$

which may be written in the form

$$r = \frac{h^2/\mu}{1 + (bh/\mu)\cos(\theta - \alpha)}. \tag{p}$$

Finally, introducing the notations

$$l = \frac{h^2}{\mu} \qquad \text{and} \qquad e = \frac{bh}{\mu}, \tag{q}$$

Eq. (p) takes the form

$$r = \frac{l}{1 + e \cos(\theta - \alpha)}. \tag{r}$$

This is the polar equation of a conic section having the latus rectum $2l$, eccentricity e, and its axis inclined to the x-axis by the angle α.

For $e < 1$, the conic section represented by Eq. (r) is an ellipse, which is, of course, the case for all planets in the solar system. For $e > 1$, the curve is a hyperbola; and if there ever were any planets in our solar system for which the initial conditions of motion were such as to make $e > 1$, they have long since escaped the attraction of the sun. For the particular case where $e = 1$, the curve is a parabola. Thus we conclude that each planet in our solar system moves around the sun in an elliptical orbit, the sun occupying one focus of the ellipse. This is the first of three laws of planetary motion announced by Kepler in 1609.

For the particular case of an ellipse, it is convenient to make, in Eq. (r), the substitution

$$l = a(1 - e^2), \tag{s}$$

where a is the major semiaxis of the ellipse and e is the eccentricity. Then Eq. (r) becomes

$$r = \frac{a(1 - e^2)}{1 + e \cos(\theta - \alpha)}. \tag{52}$$

In the case of the earth's orbit, the determination of the eccentricity

e is a comparatively simple matter. We see from Eq. (52) that the distance r between the earth and the sun varies inversely as the factor $1 + e\cos(\theta - \alpha)$. Denoting then by D_1 and D_2 the greatest and least apparent diameters of the sun when the earth has, respectively, the positions A and B on its orbit (Fig. 63), for which

$$\theta - \alpha = 0 \quad \text{and} \quad \theta - \alpha = \pi,$$

we may write

$$\frac{D_1}{1 + e} = \frac{D_2}{1 - e},$$

from which

$$e = \frac{D_1 - D_2}{D_1 + D_2}. \qquad (t)$$

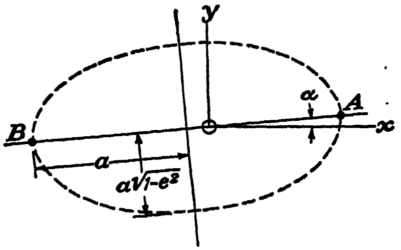


FIG. 63.

The apparent diameter of the sun can be most conveniently expressed in terms of the angle subtended by its disk and observations give $D_1 = 32' \; 36''$ on Dec. 21 and $D_2 = 31' \; 22''$ on June 21. Using these values in Eq. (t), we find, for the eccentricity of the earth's orbit, $e = \frac{1}{60}$. This is a small eccentricity and indicates that the orbit is very nearly circular, much more so, in fact, than indicated in Fig. 63.

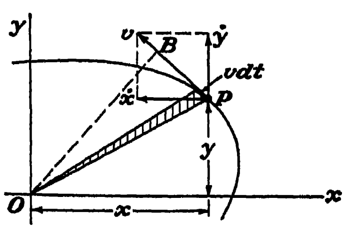


FIG. 64.

We return now to expression (e), which was obtained by taking the difference between the two original equations of motion (c). It was already used in transforming the equations of the hodograph into the equation of the orbit of the planet, but we shall now see that it holds still further significance. Noting that the left-hand side of Eq. (e) is simply the algebraic sum of moments of the components of the velocity vector v of the planet with respect to the origin O and referring to Fig. 64, we write

$$v \cdot \overline{OB} = \text{const.} = h. \qquad (53)$$

Then noting further that $v\,dt$ is a small element of the orbit and that $\frac{1}{2} \cdot v\,dt \cdot \overline{OB}$ is the area of the shaded sector swept out by the radius vector OP in time dt, we conclude from Eq. (53) that the planet moves in such manner that its radius vector sweeps out equal areas in equal intervals of time. This statement is the second of Kepler's three laws of planetary motion and is called the *law of conservation of areas.*

From the above remarks, we see that the constant h in Eq. (53) represents the doubled rate of description of area by the radius vector.

This fact enables us to determine the period of the planet, *i.e.*, the time required to make one complete trip around the sun. Since the whole area of the ellipse is swept out by the radius vector in the period τ and since $h/2$ is the constant rate of description of area, we may write

$$\tau \frac{h}{2} = \pi a a \sqrt{1 - e^2}, \tag{u}$$

where $a\sqrt{1 - e^2}$ is the minor semiaxis of the elliptical orbit. From the first of notations (q) and notation (s), we have

$$h = \sqrt{\mu l} = \sqrt{\mu a(1 - e^2)}. \tag{v}$$

Substituting this value of h into Eq. (u) and then solving for the period τ, we obtain

$$\tau = 2\pi \sqrt{\frac{a^3}{\mu}}. \tag{54}$$

Taking the constant μ to be the same for all the different planets having orbits with major semiaxis a_1, a_2, a_3, . . . and periods τ_1, τ_2, τ_3, . . . , it follows from formula (54) that

$$\frac{\tau_1^2}{a_1^3} = \frac{\tau_2^2}{a_2^3} = \frac{\tau_3^2}{a_3^3} = \cdots = \frac{4\pi^2}{\mu}. \tag{55}$$

That is, the squares of the periods of the different planets are proportional to the cubes of their mean distances from the sun. This statement is Kepler's third law of planetary motion.

If we take the earth's mean distance from the sun as unit distance and determine, by observation, the periods of the other planets, the third law enables us to calculate the mean distances of the other planets from the sun. Table IX below gives the periods of some of the planets in days and their mean distances from the sun in terms of the earth's mean distance of about 92,600,000 miles as a unit.

TABLE IX*

Planet	Period, days	Mean distance
Mercury	87.9692	0.38710
Venus	224.6176	0.72333
Earth	365.2565	1.00000
Mars	686.9785	1.52369
Jupiter	4,332.514	5.20096
Saturn	10,759.275	9.54006

* Taken from Lamb's "Dynamics," Cambridge, Eng., 1926.

In all of the preceding discussion, we have assumed that the sun was a fixed point. We know, however, that a planet attracts the sun with a force equal and opposite to that with which the sun attracts the planet. Thus the sun also must have some acceleration toward the planet. To take account of this acceleration, we can continue to use the sun as our origin of coordinates but work with the relative acceleration between the two bodies instead of the absolute acceleration of the planet. Referring to Eq. (a), we see that the force of attraction between the sun of mass M and any given planet of mass m is kMm/r^2 at the distance r. Thus the acceleration of the planet toward the sun must be kM/r^2, while the acceleration of the sun toward the planet is km/r^2. This makes the relative acceleration between the two bodies $k(M + m)/r^2$; and if we choose to use the sun as the origin of reference, we now have to take $\mu = k(M + m)$ instead of $\mu = kM$ in all of our preceding equations. As long as we are discussing the motion of any one planet relative to the sun, this change in the definition of μ does not affect any of our previous conclusions and is of no consequence. However, when we compare the observed (and hence relative) motions of different planets, the quantity μ is not quite the same for each planet, and Kepler's third law, as represented by Eq. (55), requires a small correction as follows: First, Eq. (54), for the period of a planet, becomes

$$\tau = 2\pi\sqrt{\frac{a^3}{k(M + m)}};\tag{54a}$$

and then, since $\mu = k(M + m)$ varies slightly for the different planets, we must write, instead of Eq. (55),

$$\frac{\tau_1^2}{a_1^3}(M + m_1) = \frac{\tau_2^2}{a_2^3}(M + m_2) = \frac{\tau_3^2}{a_3^3}(M + m_3) = \cdots = \frac{4\pi^2}{k}.\tag{55a}$$

In the case of the smaller planets, this correction is insignificant because m is so small compared with M; but for the larger planets such as Jupiter (for which $M/m = 1{,}047$), it can affect slightly the computed value of the mean distance a in Table IX. For example, the corrected mean distance for Jupiter is 5.2028 instead of 5.20096 as given in the table.

Formula (54a) above can be used also to calculate the mass m of a planet if it has a satellite whose period and mean distance from the planet are known. To show this, we denote by s the mass of the satellite and by τ_1 and a_1 its period and mean distance, respectively. Then applying formula (54a) first to the planet and its satellite and then to the sun and the planet, we write

$$\tau_1 = 2\pi\sqrt{\frac{a_1^3}{k(m + s)}} \quad \text{and} \quad \tau_0 = 2\pi\sqrt{\frac{a_0^3}{k(M + m)}},\tag{w}$$

where τ_0 and a_0 are the period and mean distance of the planet with respect to the sun. From these two expressions, we get

$$\frac{\tau_1^2}{\tau_0^2} = \frac{a_1^3}{a_0^3}\frac{M+m}{m+s}. \tag{x}$$

Neglecting s in comparison with m, this gives

$$m = \frac{\tau_0^2 a_1^3 M}{\tau_1^2 a_0^3 - \tau_0^2 a_1^3} \tag{y}$$

for the mass of the planet.

PROBLEMS

37. Prove that Eq. (53) holds for any plane motion of a particle under the action of a central force, *i.e.*, a force that is always directed toward a fixed point in space.

38. Prove that if the earth were suddenly deprived of its orbital velocity, it would begin to fall into the sun, which it would reach in about 65 days.

39. Compute the average velocity in miles per hour with which the earth travels around the sun. Assume a circular orbit. *Ans.* $v = 67{,}200$ m.p.h.

40. The moon makes 13.369 trips around the earth in one year. The ratio of the earth's mean distance from the sun to the moon's mean distance from the earth is 388.86. Using formula (y), compute the ratio M/m of the mass of the sun to the mass of the earth. *Ans.* $M/m = 329{,}000$.

12. Motion of a Projectile.—The study of the motion of projectiles through the air is generally known as the science of *exterior ballistics.* In discussing this problem, we shall first develop the general ballistics

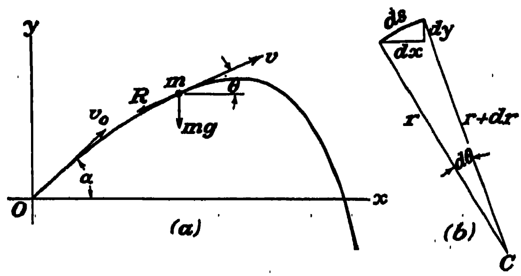


FIG. 65.

equations considering the projectile as a particle and then illustrate their solution for several particular cases. Referring to Fig. 65a, which represents the *trajectory*, or path, of a projectile shot from point O with initial velocity v_0 and angle of elevation α, we choose rectangular coordinate axes x and y directed as shown. Considering the projectile at any point along its path, we see that it will be acted upon by two forces: (1) the vertical gravity force mg (assumed constant) and (2) the air resistance R, which is always tangent to the path and opposes the motion. As already discussed in Art. 4, we take this resistance in the general form

$$R = -Ckf(v), \tag{a}$$

where k is a physical constant depending on air density and size of the projectile and C is a factor called the *coefficient of resistance*. Projecting these forces onto the x- and y-axes, the equations of motion become

$$\left.\begin{aligned} m\ddot{x} &= -Ckf(v)\cos\theta, \\ m\ddot{y} &= -Ckf(v)\sin\theta - mg, \end{aligned}\right\} \tag{b}$$

where m is the mass of the projectile and θ is the angle of inclination of the tangent to the path at any point. Introducing the notation

$$c = \frac{Ck}{m}, \tag{c}$$

sometimes called the *ballistics coefficient*, Eqs. (b) may be written

$$\left.\begin{aligned} \ddot{x} &= -cf(v)\cos\theta, \\ \ddot{y} &= -cf(v)\sin\theta - g. \end{aligned}\right\} \tag{56}$$

These are the differential equations of motion of a projectile in rectangular coordinates. Noting that $\dot{x} = v\cos\theta$ and $\dot{y} = v\sin\theta$, they can also be written in the form

$$\left.\begin{aligned} \frac{d}{dt}(v\cos\theta) &= -cf(v)\cos\theta, \\ \frac{d}{dt}(v\sin\theta) &= -cf(v)\sin\theta - g. \end{aligned}\right\} \tag{57}$$

In a few particular cases, Eqs. (56) or (57) can be integrated directly. As an example, we take the case where the resistance is proportional to the velocity; *i.e.*, $cf(v) = cv$, and c is a constant. Then Eqs. (56) become

$$\ddot{x} = -cv\cos\theta, \qquad \ddot{y} = -cv\sin\theta - g,$$

which we write in the form

$$\ddot{x} + c\dot{x} = 0, \qquad \text{and} \qquad \ddot{y} + c\dot{y} = -g. \tag{d}$$

These are independent linear differential equations with constant coefficients which can be integrated without difficulty by the method used in Arts. 5 and 6. The general solutions are

$$\left.\begin{aligned} x &= C_1 + C_2e^{-ct}, \\ y &= D_1 + D_2e^{-ct} - \frac{gt}{c}, \end{aligned}\right\} \tag{e}$$

where C_1, C_2, D_1, D_2 are constants of integration. To evaluate these constants, we have the initial conditions

$$(x)_{t=0} = 0, \qquad (y)_{t=0} = 0, \qquad (\dot{x})_{t=0} = \dot{x}_0 = v_0\cos\alpha, \qquad (\dot{y})_{t=0} = \dot{y}_0 = v_0\sin\alpha.$$

Using these conditions in Eqs. (e) and their first derivatives with respect to time, we find

$$C_1 = \frac{\dot{x}_0}{c} = -C_2,$$
$$D_1 = \frac{\dot{y}_0 + g/c}{c} = -D_2$$

and Eqs. (e), in final form, become

$$\left.\begin{aligned} x &= \frac{\dot{x}_0}{c}(1 - e^{-ct}),\\ y &= \frac{\dot{y}_0 + g/c}{c}(1 - e^{-ct}) - \frac{gt}{c} \end{aligned}\right\} \tag{58}$$

The general nature of the trajectory defined by these equations is shown in Fig. 66. From the first equation, we see that the trajectory approaches asymptotically to the vertical $x = \dot{x}_0/c$; while from the second equation, we see that the projectile finally tends to fall vertically with the terminal velocity $V = g/c$.

If we take c very small, we approach the case of motion of the projectile without air resistance. In such case, we may use, for e^{-ct}, only the first three terms of the series

$$e^{-ct} = 1 - ct + \frac{(ct)^2}{2} - \cdots,$$

and Eqs. (58) become

$$x = \dot{x}_0 t - \frac{\dot{x}_0 c t^2}{2},$$
$$y = \dot{y}_0 t - \frac{gt^2}{2} - \frac{\dot{y}_0 c t^2}{2}.$$

Then taking $c = 0$ in these expressions, we obtain, for a projectile without resistance

$$x = \dot{x}_0 t, \qquad y = \dot{y}_0 t - \frac{gt^2}{2}. \tag{f}$$

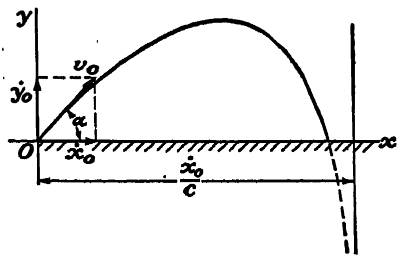


FIG. 66.

Finally, eliminating the parameter t, we obtain for the trajectory the known parabola

$$y = \frac{\dot{y}_0}{\dot{x}_0}x - \frac{g}{2\dot{x}_0^2}x^2. \tag{59}$$

If the resistance $R = -Ckf(v)$ is taken proportional to some power of the velocity, we must proceed in a somewhat less direct manner than that used above. In all such cases, it will be helpful to write, in addition to Eqs. (57) in rectangular coordinates, the equations of motion in natural coordinates, *i.e.*, in the tangential and normal directions, respectively. Referring again to Fig. 65*a* and projecting the acting forces onto the tangent and normal to the path at any point, we write

$$\left.\begin{aligned} \frac{dv}{dt} &= -cf(v) - g\sin\theta,\\ \frac{v^2}{r} &= g\cos\theta, \end{aligned}\right\} \tag{60}$$

where dv/dt is the *tangential acceleration* and v^2/r is the *normal acceleration*, r being the radius of curvature of the path at any point.

The second of Eqs. (60) will be especially useful to us owing to the fact that it is independent of the resistance function $-cf(v)$ and will apply equally well to all cases. This equation can be written in several different forms, all of which will be helpful in our further discussion. Recalling that the curvature[1]

$$\frac{1}{r} = -\frac{d^2y/dx^2}{[1 + (dy/dx)^2]^{\frac{3}{2}}},$$

wherein $dy/dx = \tan\theta$ and $1 + \tan^2\theta = 1/\cos^2\theta$, we have

$$\frac{1}{r} = -\frac{d^2y}{dx^2}\cos^3\theta.$$

In this way, the second of Eqs. (60) becomes

$$\frac{d^2y}{dx^2} = -\frac{g}{v^2\cos^2\theta}. \tag{61}$$

Again, referring to Fig. 65*b*, we see that $1/r = -d\theta/ds$, and the same equation takes the form

$$v^2\frac{d\theta}{ds} = -g\cos\theta. \tag{62}$$

Finally, since $v = ds/dt$, Eq. (62), in turn, can be written

$$\frac{d\theta}{dt} = -\frac{g\cos\theta}{v}. \tag{63}$$

Equations (61), (62), and (63) are the three forms in which the second of Eqs. (60) will be useful.

We return now to the first of Eqs. (57). Multiplying both sides of this equation by $d\theta$, we obtain

$$d(v\cos\theta)\frac{d\theta}{dt} = -cf(v)\cos\theta\, d\theta.$$

Then replacing $d\theta/dt$ by its value (63), we get

$$gd(v\cos\theta) = vcf(v)d\theta. \tag{64}$$

This is a first-order differential equation defining the relation between the magnitude v of the velocity vector and its inclination θ.

If, for each point along the trajectory in Fig. 65*a*, we lay out, from a fixed origin O, the corresponding velocity vector v of the projectile, the

[1] The minus sign is taken to agree with our coordinate system in Fig. 65.

locus of the ends of these vectors defines the *hodograph*[1] as shown in Fig. 67. We see that Eq. (64) is the differential equation of this hodograph in polar coordinates.

It can be shown in a general way that once we have the solution of Eq. (64), *i.e.*, once we know the relation between v and θ, the path of the projectile can be completely defined by quadratures. For this purpose, we have, from Eqs. (62) and (63), the following expressions:

$$ds = -\frac{v^2 d\theta}{g \cos \theta}, \quad \cdots \quad (65) \qquad dt = -\frac{v\, d\theta}{g \cos \theta}, \quad \cdots \quad (66)$$

$$dx = ds \cos \theta = -\frac{v^2 d\theta}{g}, \qquad (67)$$

$$dy = dx \tan \theta = -\frac{v^2}{g} \tan \theta \, d\theta \quad \cdots \quad . \qquad (68)$$

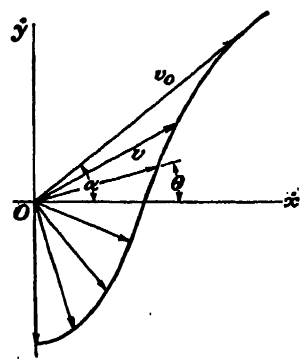


FIG. 67.

The solution of Eq. (64) gives us $v = f(\theta)$, after which the quantities s, t, x, y defining the trajectory can all be expressed by integrals involving θ as a parameter. Thus Eqs. (64) and (65) to (68) represent a general approach to the ballistics problem.

To illustrate, let us consider first the very simple case of a projectile without air resistance, for which Eq. (64) reduces to

$$d(v \cos \theta) = 0.$$

Integrating this, we get

$$v \cos \theta = \text{const.} = v_0 \cos \alpha,$$

from which

$$v = \frac{v_0 \cos \alpha}{\cos \theta}. \qquad (g)$$

From this simple relation between v and θ, we see that the hodograph is a straight line AB as shown in Fig. 68. Using expression (g) in Eqs. (67) and (68) above and remembering that $(x)_{\theta=\alpha} = (y)_{\theta=\alpha} = 0$, we obtain

$$\left.\begin{aligned} x &= -\frac{v_0^2 \cos^2 \alpha}{g} \int_\alpha^\theta \frac{d\theta}{\cos^2 \theta} = \frac{v_0^2 \cos^2 \alpha}{g} (\tan \alpha - \tan \theta), \\ y &= -\frac{v_0^2 \cos^2 \alpha}{g} \int_\alpha^\theta \frac{\sin \theta \, d\theta}{\cos^3 \theta} = \frac{v_0^2 \cos^2 \alpha}{2g} (\sec^2 \alpha - \sec^2 \theta). \end{aligned}\right\} \qquad (h)$$

These equations define the trajectory in terms of θ as a parameter. Eliminating θ through the relationship $1 + \tan^2 \theta = \sec^2 \theta$, we obtain

[1] See the authors' "Engineering Mechanics," 2d ed., p. 345.

$$y = x \tan \alpha - \frac{gx^2}{2v_0^2 \cos^2 \alpha}, \tag{69}$$

which, of course, agrees with Eq. (59) above.

As a major example, we now assume that the resistance is proportional to the square of the velocity and take $cf(v) = cv^2$, where c is a constant. As already noted in Art. 4, this quadratic law of resistance holds with good accuracy for many shapes of projectiles, provided the velocity v is less than the speed of sound (1,120 f.p.s.). For such resistance, Eq. (64) becomes

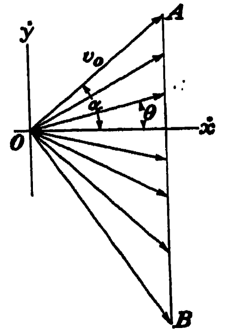


FIG. 68.

$$gd(v \cos \theta) = cv^3 d\theta. \tag{i}$$

Instead of integrating this equation directly, we shall find it convenient first to make a simple transformation. Taking $\dot{x} = v \cos \theta$ as a new variable and using Eq. (65), we can write Eq. (i) in the form

$$\frac{d\dot{x}}{\dot{x}} = \frac{c}{g} \frac{v^2 d\theta}{\cos \theta} = -c\, ds. \tag{j}$$

Then integrating Eq. (j) and using the initial condition $(\dot{x})_{s=0} = \dot{x}_0$, we obtain

$$\ln \dot{x} - \ln \dot{x}_0 = -cs$$

or

$$\dot{x} = \dot{x}_0 e^{-cs}. \tag{k}$$

Finally, using this expression in Eq. (61), we obtain

$$\frac{d^2y}{dx^2} = -\frac{g}{\dot{x}^2} = -\frac{g}{\dot{x}_0^2} e^{2cs}. \tag{l}$$

In the case of a flat trajectory (Fig. 69a) such as we might have for a rifle bullet or archer's arrow, a good approximate solution of Eq. (l) can be made if we replace s by x. In this way, the equation becomes

$$\frac{d^2y}{dx^2} = -\frac{g}{\dot{x}_0^2} e^{2cx}, \tag{m}$$

which can be integrated without further difficulty. A first integration gives

$$\frac{dy}{dx} = -\frac{ge^{2cx}}{2c\dot{x}_0^2} + C_1.$$

Observing that $(dy/dx)_{x=0} = \tan \alpha$, we find

$$C_1 = \tan \alpha + \frac{g}{2c\dot{x}_0^2}.$$

Integrating again,

$$y = -\frac{ge^{2cx}}{4c^2\dot{x}_0^2} + C_1x + C_2,$$

and from the initial condition $(y)_{x=0} = 0$,

$$C_2 = \frac{g}{4c^2\dot{x}_0^2}.$$

Thus the equation of the path becomes

$$y = \frac{g}{4c^2\dot{x}_0^2}(1 - e^{2cx}) + \left(\tan\alpha + \frac{g}{2c\dot{x}_0^2}\right)x, \tag{70}$$

which can also be written in the form

$$y = x\tan\alpha - \frac{gx^2}{2v_0^2\cos^2\alpha}\left(\frac{e^z - z - 1}{\frac{1}{2}z^2}\right), \tag{71}$$

where $z = 2cx$.

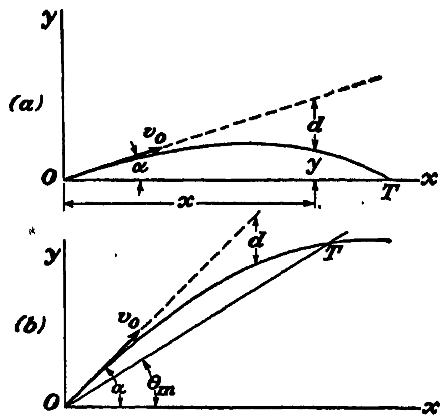


FIG. 69.

Comparing Eq. (71) with Eq. (69), we see that for quadratic resistance, the approximate equation (71) of the trajectory has the form of a modified parabola, the deviation therefrom depending on the quantity

$$F(z) = \frac{e^z - z - 1}{\frac{1}{2}z^2} = 1 + \frac{z}{3} + \frac{z^2}{4\times 3} + \frac{z^3}{5\times 4\times 3} + \cdots. \tag{n}$$

We also note, from Fig. 69a, that the last term in expression (71) represents the drop d of the projectile away from its initial tangent. Thus, the formula

$$d = \frac{gx^2}{2v_0^2\cos^2\alpha}F(z) \tag{72}$$

can be used to compute this important quantity.

Example: An arrow having initial horizontal velocity $v_0 = 235$ f.p.s. weighs 0.033 lb. and has a diameter $D = 0.02$ ft. Assuming a coefficient of resistance $C = 1.00$ and taking air density $\rho = 0.00237$ slug per cu. ft., we obtain, for the ballistics coefficient,

$$c = \frac{C\frac{1}{2}\rho a}{m} = \frac{0.3723(10)^{-6}}{1.026(10)^{-3}} = 0.3635(10)^{-3}\text{ ft.}^{-1}$$

Then for a range of 30 yd. = 90 ft., we have $z = 2cx = 0.0654$ and using the series (n) above, $F(z) = 1.0222$. Thus, by Eq. (72), the drop of the arrow in 30 yd. is

$$d = \frac{32.2(90)^2}{2(235)^2}(1.0222) = 2.41\text{ ft.}$$

Although we assumed a flat trajectory in the development of Eqs. (71) and (72), they can also be used for the case shown in Fig. 69b by taking

$$z = 2c\frac{x}{\cos\theta_m} = 2\beta cx, \tag{o}$$

where $\beta = 1/\cos\theta_m$, θ_m being the inclination of the chord OT with the horizontal. With this more general definition of z, Eqs. (71) and (72) will apply with good accuracy to any portion of the trajectory that does not deviate too much from its own chord.[1]

In the case of large-range trajectories where the approximation involved in Bernoulli's solution will no longer be permissible, we must proceed directly with the integration of Eq. (i) without making the transformation represented by Eq. (j). For this purpose, we write Eq. (i) in the form

$$\frac{d(v\cos\theta)}{(v\cos\theta)^3} = -\frac{c}{g}\frac{d\theta}{\cos^3\theta}. \tag{p}$$

Then integrating, we have

$$-\frac{1}{v^2\cos^2\theta} + \frac{1}{v_0^2\cos^2\alpha} = -\frac{2c}{g}\int_\alpha^\theta \frac{d\theta}{\cos^3\alpha} = -\frac{2c}{g}[\xi(\theta) - \xi(\alpha)], \tag{q}$$

where

$$\xi(\theta) = \int_0^\theta \frac{d\theta}{\cos^3\theta} = \frac{1}{2}\left(\frac{\sin\theta}{\cos^2\theta} + \frac{1}{2}\ln\frac{1+\sin\theta}{1-\sin\theta}\right). \tag{73}$$

With a little algebraic manipulation, Eq. (q) can be put in the more convenient form

$$v^2\cos^2\theta = \frac{v_0^2}{\sec^2\alpha + \frac{2v_0^2c}{g}[\xi(\theta) - \xi(\alpha)]}. \tag{74}$$

[1] This approximate solution of the ballistics problem with quadratic resistance was first made by John Bernoulli (1719).

This is the polar equation of the hodograph, which can now be constructed for any particular case, *i.e.*, for given values of v_0, α, and c. The general form of this hodograph has already been shown in Fig. 67.

The quantity $2v_0^2c/g$ in Eq. (74) has a simple physical significance. Recalling that for our assumption of quadratic resistance,

$$R = Ckv^2 = mcv^2$$

is the magnitude of the resisting force and setting this equal to the gravity force mg, we obtain, for the so-called *terminal velocity*,

$$V = \sqrt{\frac{g}{c}}, \tag{r}$$

as already discussed in Art. 4, page 28. Thus

$$\frac{2v_0^2c}{g} = 2\left(\frac{v_0}{V}\right)^2 \tag{s}$$

is a dimensionless quantity depending only on the ratio v_0/V of initial velocity to terminal velocity of the projectile.

After the hodograph has been constructed so that we have a series of values of v^2 for chosen values of θ between α and $-\pi/2$, the trajectory itself may be constructed by using Eqs. (67) and (68), from which

$$x = -\int_\alpha^\theta \frac{v^2}{g}\,d\theta \qquad \text{and} \qquad y = -\int_\alpha^\theta \frac{v^2}{g}\tan\theta\,d\theta. \tag{75}$$

These quadratures must be made by graphical or numerical integration as discussed in Arts. 2 and 3.

The foregoing solution based on the assumption of true quadratic resistance applies with good accuracy to projectiles moving with velocities less than the velocity of sound, *e.g.*, bombs dropped from aircraft. In the case of projectiles having high initial speeds, we can continue to use the quadratic law only if we represent the coefficient of resistance C as a function of velocity as shown by the curve in Fig. 19, page 26, for a small-bore rifle bullet. In this case, Eq. (64) must be integrated by some approximate procedure. A graphical method well suited to the purpose will now be described.

Taking $\dot{x} = v\cos\theta$ as our dependent variable and letting

$$vf(v) = v^3 = \left(\frac{\dot{x}}{\cos\theta}\right)^3,$$

we write Eq. (64) in the form

$$\frac{d\dot{x}}{d\theta} = \frac{c}{g}v^3 = \frac{c}{g}\left(\frac{\dot{x}}{\cos\theta}\right)^3, \tag{76}$$

in which $c = Ck/m$ is no longer a constant. To integrate Eq. (76)

graphically, we first draw, in the $\dot{x}\theta$-plane, a family of curves

$$\dot{x} = v \cos \theta$$

for a series of constant values of v chosen within the probable range of variation of the velocity of the projectile. In our case, these curves, called *isoclines*, are simple cosine curves with various amplitudes v as shown in Fig. 70. We see from Eq. (76) that each isocline represents the locus of points in the $\dot{x}\theta$-plane for which the family of integral curves defined by Eq. (76) all have the same slope $(c/g)v^3$. Our problem is to draw the particular integral curve that passes through the initial point A ($\dot{x} = v_0\cos\alpha$ when $\theta = \alpha$) with the known slope $(c_0/g)v_0^3$ and crosses each isocline with the corresponding slope $(c/g)v^3$ associated therewith.

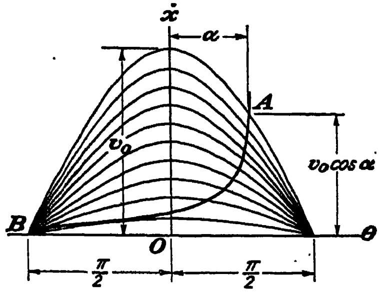


FIG. 70.

Example: To illustrate the graphical integration of Eq. (76) by the isocline method, we take the case of a 150-grain 30-caliber rifle bullet fired with muzzle velocity $v_0 = 2{,}380$ f.p.s. and angle of elevation $\alpha = 45$ deg. Then assuming constant air density $\rho = 0.00237$ slug per cu. ft., we have

$$\frac{c}{g} = \frac{C(\frac{1}{2}\rho a)}{mg} = 27(10)^{-6}C \text{ sec.}^2/\text{ft.}^2 \qquad (t)$$

where the coefficient of resistance C, which now varies with the velocity, is to be taken from the curve in Fig. 19, page 26.

We begin with a suitable series of arbitrary values of v as shown in column (1) of Table X and construct the corresponding isoclines $\dot{x} = v \cos \theta$ as shown in Fig. 71. For each value of v, the corresponding value of C, taken from Fig. 19, is shown in column (3), and the value of c/g, computed from expression (t), is recorded in column (4). Finally, the slopes $(c/g)v^3$ associated with the various isoclines are shown in column (5). Using these computed slopes, the series of rays shown at the left of Fig. 71 are constructed.

These preliminaries finished, we are ready to begin the construction of the required integral curve $\dot{x} = f(\theta)$. Since $\dot{x}_0 = v_0\cos\alpha = 1{,}680$ f.p.s. and $\theta_0 = \alpha = 45$ deg., we start at point A with the slope $(d\dot{x}/d\theta)_0 = 127{,}200$ f.p.s. per rad. as represented by the steepest ray in Fig. 71 and draw the line $A1$. At point b on this line, midway between the isoclines $v = 2{,}380$ and $v = 2{,}000$, we start the next segment $b2$ parallel to the second ray and continue in this way until we intersect the isocline $v = 800$ at point B. The smooth curve inscribed in the polygon $AbcdefghiB$ and tangent thereto at the points where it crosses the isoclines represents the required integral curve $\dot{x} = f(\theta)$; but to avoid confusion of lines in the drawing, we have omitted the smooth curve and shown only the polygon.

At point B, we see that the projectile has a velocity of 800 f.p.s. inclined to the horizontal by the angle $\theta = 41° 30'$. From this point on, we may take $v_1 = 800$ f.p.s. and $\alpha_1 = 41° 30'$ as new initial values and continue the solution numerically on the basis of Eq. (74) for true quadratic resistance.

TABLE X

(1) v ft./sec.	(2) $(10)^{-6}v^3$ ft.^{3}/sec.^{3}	(3) C (see Fig. 19)	(4) $(10)^6 \frac{c}{g}$ sec.^{2}/ft.^{2}	(5) $\frac{c}{g} v^3$ ft./sec./rad.
2,380	13,481	0.350	9.45	127,200
2,000	8,000	0.395	10.66	85,400
1,600	4,096	0.445	12.00	49,100
1,400	2,744	0.468	12.64	34,700
1,200	1,728	0.450	12.14	21,000
1,100	1,331	0.370	10.00	13,300
1,000	1,000	0.220	5.94	5,940
900	729	0.185	5.00	3,640
800	512	0.180	4.86	2,490

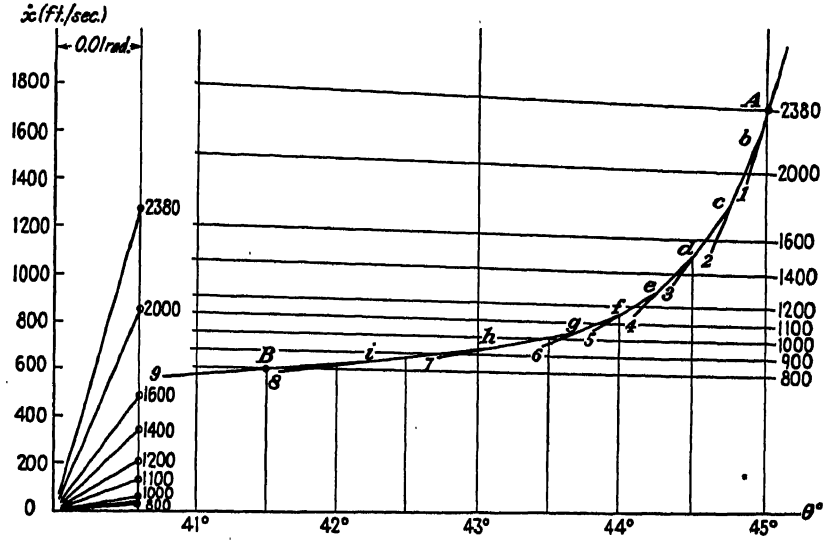


FIG. 71.

To find the coordinates x_1 and y_1 of the trajectory corresponding to point B in Fig. 71, we must integrate Eqs. (75) by graphical or numerical quadrature. Using Simpson's rule, a numerical integration for x together with the final results for y are shown in Table XI. We see that the projectile has already reached a height of almost $\frac{1}{2}$ mile.

In the foregoing discussion, we have tried to illustrate a few of the many methods of approach that have been developed for treating the problem of exterior ballistics.[1] Naturally the problem is a highly special-

[1] For a resumé of the most important methods of approach to this problem, see article by Otto von Eberhardt, "Handbuch der physikalischen und technischen Mechanik," Leipzig, vol. II, p. 182, 1930.

ized one, and we have not considered several important factors such as the variation of c with altitude, drift due to gyroscopic effect, etc. When all these factors are taken into account, the equations of motion are extremely complex and some kind of mechanical analyzer will be used in their solution.

TABLE XI

θ deg.	$\dot{x}$ ft./sec.	$\cos\theta$	v ft./sec.	$(10)^{-6}v^2$ ft.^{2}/sec^2.	$\eta = \frac{v^2}{g}$ ft.	$\eta_1 + 4\eta_2 + \eta_3$	Δx ft.	x ft.	y ft.
45–00	1,680	0.707	2,380	5.66	176,000	....	0	0	0
44–45	1,310	0.710	1,845	3.40	105,600				
44–30	1,080	0.713	1,515	2.29	71,100	669,500	974	974	967
44–00	840	0.719	1,168	1.36	42,200				
43–30	740	0.725	1,020	1.04	32,300	272,200	792	1,766	1,733
43–00	690	0.731	944	0.891	27,700				
42–30	655	0.737	889	0.790	24,500	167,600	488	2,254	2,187
42–00	625	0.743	842	0.709	22,000				
41–30	600	0.749	800	0.640	19,900	132,400	385	2,639	2,534

From a practical point of view, most problems of fire control in the field will be solved with the aid of ballistics tables.[1] For a desired range and a given muzzle velocity, such tables usually give directly the required angle of elevation, the maximum height, the distance and time of travel, the angle of fall at the target, etc. In addition, formulas and tables for correcting for wind deflection, drift due to gyroscopic effect, variation in air density with temperature and altitude, etc., are also given.

PROBLEMS

41. Using Eqs. (*h*), page 98, show that the maximum height h attained by a projectile without air resistance is

$$h = \frac{v_0^2}{2g}\sin^2\alpha$$

while the range l is

$$l = \frac{v_0^2}{g}\sin 2\alpha.$$

42. Using Eq. (72) and assuming a constant coefficient of resistance $C = 0.4$, compute the drop d of a 150-grain 30-caliber rifle bullet with horizontal muzzle velocity $v_0 = 2{,}380$ f.p.s. at a range of 300 yd.

43. Construct, in polar coordinates, the hodograph for a projectile with true viscous resistance, *i.e.*, proportional to the first power of the velocity.

44. Using the data in Tables X and XI and Eq. (66), compute the time t required for the projectile to reach the point $x_1 = 2{,}639$ ft., $y_1 = 2{,}534$ ft. in its trajectory.

45. Using the data in Tables X and XI and Eq. (74) for the hodograph, continue the trajectory calculations for the 30-caliber rifle bullet discussed in the example above and determine (*a*) the maximum height h and (*b*) the range l.

[1] See, for example, Ingalls, "Ballistic Tables," Government Printing Office, Washington, D.C., 1918.

CHAPTER II

DYNAMICS OF A SYSTEM OF PARTICLES

13. Principle of Linear Momentum.—In Chap. I, we have already discussed the dynamics of a particle and considered the application thereof to various engineering problems. We proceed now to the case of a system of particles where, in addition to external forces, we must consider also the effect of internal forces, representing the actions and reactions between particles. Our solar system is a good example. As internal forces, we have the attractions between planets and between the planets and the sun; as external forces, we have the action of fixed stars not included in our solar system. In this case, there is nothing to interfere with free motion of the planets, and we have a *system of particles without constraints.* In engineering applications, we shall often encounter systems of particles that are not free in their relative motions but are subjected to various physical constraints such as bearings, guides, cams, etc., so that we have a *system of particles with constraints.* Sometimes the constraints are such that they bring the particles into a rigid system. In writing equations of dynamics for a physical body, we can usually neglect its elasticity and treat it as a *rigid body;* the distances between particles are assumed invariable. Most machines and structures, in turn, can be treated as systems of rigid bodies connected between themselves and the foundation by such constraints as hinges, bearings, guides, etc.

Let us consider now any system of particles with masses m_1, m_2, m_3, . . . having velocities v_1, v_2, v_3, . . . , respectively. The quantity mv for any one of these particles is its *linear momentum,* which can be represented by a vector having the direction of the velocity. Making the geometrical sum of the linear momenta of all particles, we obtain the *linear momentum of the system.* In our further discussion, we shall usually resolve the velocity of each particle into orthogonal components $\dot{x}$, $\dot{y}$, $\dot{z}$; then the corresponding components of the linear momentum of the system will be

$$\Sigma m\dot{x}, \quad \Sigma m\dot{y}, \quad \Sigma m\dot{z},$$

where the summations are understood to include all particles of the system.

For each particle of a system, we can write three equations of motion as follows:

$$\left.\begin{aligned} \frac{d}{dt}(m\dot{x}) &= X_e + X_i, \\ \frac{d}{dt}(m\dot{y}) &= Y_e + Y_i, \\ \frac{d}{dt}(m\dot{z}) &= Z_e + Z_i, \end{aligned}\right\} \qquad (a)$$

where X_e, Y_e, Z_e denote the projections of the resultant external force acting on any one particle and X_i, Y_i, Z_i the projections of the resultant internal force for that particle. Making a summation of such equations of motion for all particles of a system, we obtain

$$\left.\begin{aligned} \frac{d}{dt}\sum(m\dot{x}) &= \sum X_e, \\ \frac{d}{dt}\sum(m\dot{y}) &= \sum Y_e, \\ \frac{d}{dt}\sum(m\dot{z}) &= \sum Z_e. \end{aligned}\right\} \qquad (77)$$

The summation of internal forces vanishes in each case, since, from the law of action and reaction, these forces always appear as pairs of balanced colinear forces which cancel each other in the summation process. Equations (77) state that the rate of change of the projection of the linear momentum of a system of particles on any axis is equal to the sum of projections on the same axis of all external forces acting on the system. These three equations represent the *principle of linear momentum.*

If the linear momentum of a system is represented by a vector, the rate of change of this vector during motion of the system will be represented by the velocity of its end point. Thus, on the basis of Eqs. (77), we conclude that the velocity of the end of the linear momentum vector is identical with the geometric sum of all external forces acting on the system. If the system is free from external forces or if their geometric sum vanishes, the linear momentum vector remains constant in magnitude and direction regardless of the relative motions between the various particles.

Equations (77) can be written in another form by introducing the familiar notion of *mass-center.* The coordinates of this point are given by the equations[1]

$$x_c = \frac{\Sigma(mx)}{\Sigma m}, \qquad y_c = \frac{\Sigma(my)}{\Sigma m}, \qquad z_c = \frac{\Sigma(mz)}{\Sigma m}, \qquad (b)$$

[1] See the authors' "Engineering Mechanics," 2d ed., p. 95.

in which m denotes the mass of each infinitesimal element into which the body is subdivided and x, y, z its coordinates. Differentiating Eqs. (b) with respect to time and introducing the notation $M = \Sigma m$ for the total mass of the system, we obtain

$$M\dot{x}_c = \Sigma(m\dot{x}), \qquad M\dot{y}_c = \Sigma(m\dot{y}), \qquad M\dot{z}_c = \Sigma(m\dot{z}). \qquad (78)$$

From these expressions, we conclude that the linear momentum of a system of particles is the same as if the total mass were concentrated at the mass-center and endowed with the velocity of that point.

Using Eqs. (78), the previous Eqs. (77) representing the principle of linear momentum for a system of particles can now be written in the form

$$\frac{d}{dt}(M\dot{x}_c) = \sum X_e, \qquad \frac{d}{dt}(M\dot{y}_c) = \sum Y_e, \qquad \frac{d}{dt}(M\dot{z}_c) = \sum Z_e. \qquad (79)$$

From these equations, we see that the mass-center moves like a particle having the total mass M of the system and acted upon by a force equal to the geometric sum of all external forces acting on the individual particles. Since the orthogonal directions x, y, z are arbitrary, we can summarize the principle of linear momentum by the following statement: *The rate of change of linear momentum in any fixed direction is equal to the resolved external force in that direction.*

Using the principle of linear momentum, we can sometimes make important conclusions regarding the motion of a system without going into a detailed consideration of the motions of its individual particles. Thus, if the system is free from external forces, we conclude that the mass-center can move only rectilinearly with a constant velocity. For example, if we neglect the actions of fixed stars on our solar system, we conclude that the mass-center of the system must move rectilinearly through space with uniform speed. Again, in the case of a shell that explodes in mid-air, we conclude, in the absence of air resistance, that the mass-center of the flying fragments continues along the same trajectory that the shell was describing before it burst. In the case of a reciprocating engine, we can likewise make the over-all observation that to avoid unbalanced disturbing forces on the foundation, the moving parts must be so proportioned and arranged that their mass-center remains stationary while the engine is running.

By the same principle, we conclude that without the help of some external force, a live system, by the creation of internal forces alone, can never bring its mass-center into motion from a state of rest. Thus, for example, walking on a smooth horizontal surface without friction will be quite impossible. Likewise, without friction which creates an

external force on its wheels, a locomotive initially at rest cannot bring itself into motion along a level track. Paradoxically, we see that friction, which we usually regard as a hinderance to motion, is the external force by which many of our so-called *self-propelled vehicles* are driven.

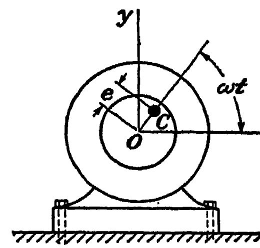


FIG. 72.

Equations (79) are especially useful in those cases where the motion of the mass-center of a system is known and it is required to find the external forces producing that motion. Consider, for example, an electric motor bolted to a rigid foundation as shown in Fig. 72, and assume that owing to inaccuracies in workmanship, the center of gravity C of the rotor does not coincide with the axis of rotation O but describes, during motion, a circle of radius e. Then measuring the angle of rotation as shown in the figure and denoting by ω the constant angular velocity of the rotor, we find

$$x_c = e \cos \omega t, \qquad y_c = e \sin \omega t. \tag{c}$$

Denoting the rotor mass by M and substituting expressions (c) in Eqs. (79), we obtain

$$-Me\omega^2\cos \omega t = \Sigma X_e, \qquad -Me\omega^2\sin \omega t = \Sigma Y_e. \tag{d}$$

These expressions represent the projections of the resultant external force exerted on the motor frame by the foundation. With reversed signs,

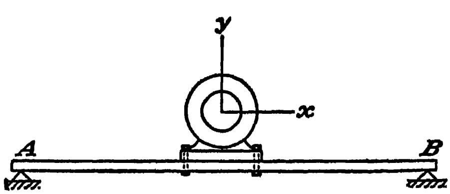


FIG. 73.

they represent the more practical concept of the disturbing forces that the unbalanced motor exerts on its foundation. The same procedure can be used in discussing the forces that an unbalanced reciprocating engine exerts on its foundation.[1]

As a second example, let us consider forced vibrations of the system in Fig. 73, representing a motor supported by an elastic beam AB. In discussing this problem, we assume that the center of gravity of the rotor has some eccentricity e as in Fig. 72 and that it runs with constant angular velocity ω. To take account of the mass of the beam during vibrations,

[1] This problem is discussed in Art. 17, p. 136.

we assume that one-half of its mass is added to the mass of the stator and denote this joint mass by W_1/g. Thereafter, we treat the beam as a spring without mass. Denoting the mass of the rotor by W_2/g and measuring vertical displacements y_1 of the stator from the position of static equilibrium, the coordinate y_2 of the center of gravity of the rotor at any instant will be

$$y_2 = y_1 + e \sin \omega t. \tag{e}$$

Thus the momentum of the system in the y-direction is

$$\frac{W_1}{g}\dot{y}_1 + \frac{W_2}{g}\dot{y}_2 = \frac{W_1 + W_2}{g}\dot{y}_1 + \frac{W_2}{g} e\omega \cos \omega t. \tag{f}$$

Considering the vertical forces acting on the system, we observe that the gravity force is balanced by the beam reactions corresponding to the static deflection, and we have to consider only the additional reactions due to displacement y_1 away from the equilibrium position. This resultant vertical external force can be taken as $-ky_1$, where k is the spring characteristic for the elastic beam. Then the principle of linear momentum gives

$$\frac{d}{dt}\left(\frac{W_1 + W_2}{g}\dot{y}_1 + \frac{W_2}{g} e\omega \cos \omega t\right) = -ky_1,$$

from which

$$\frac{W_1 + W_2}{g}\ddot{y}_1 + ky_1 = \frac{W_2}{g} e\omega^2 \sin \omega t. \tag{80}$$

This is the same equation as that obtained for a single mass particle suspended by an elastic spring and subjected to a simple harmonic disturbing force. It is worthy of note that during forced vibration of the system in Fig. 73, the actual path of the rotor center of gravity is not circular but elliptical. Nevertheless, we conclude from Eq. (80) that the disturbing force is a true simple harmonic function of time.

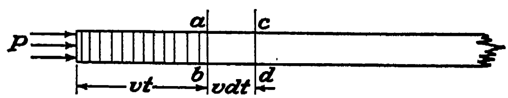


FIG. 74.

The principle of linear momentum can be used to advantage in discussing wave propagation in an elastic medium. Consider, for example, a horizontal metallic bar of uniform cross-sectional area A supported in such a way that it can move without external resistance in the horizontal direction (Fig. 74). If a compressive force P, representing a blow, is suddenly applied at one end of this bar, it produces compressive stress

P/A and corresponding compressive strain P/AE where E is the elastic modulus of the material. This state of compression will be propagated along the bar with velocity v. To find this velocity of wave propagation, we consider any instant of time t for which the wave front has progressed to the section ab and the shaded portion of the bar is in a compressed condition. After an interval of time dt, the front of this compression zone will have reached the section cd at the distance $v\,dt$ ahead of ab. We see now that the center of gravity of the compressed zone is also moving along the bar but with a velocity v_1 different from v. The velocity v_1 can be found by observing that the displacement of the compressed zone during the time dt equals the compression of the element $abcd$. Thus

$$v_1 = \frac{Pv}{AE}\frac{dt}{dt} = \frac{Pv}{AE}. \tag{g}$$

Since the compressed zone represents the total mass that is in motion, the increase in momentum of the system during the interval dt is the momentum gained by the element $abcd$, which was at rest at the instant t and has the velocity v_1 after the interval dt. This change in momentum is the mass of the element multiplied by the velocity v_1, *i.e.*,

$$\frac{Awv\,dt}{g}\frac{Pv}{AE}, \tag{h}$$

where w is the specific weight of the material. Equating the rate of change of this momentum to the external force P, we obtain

$$\frac{Pwv^2}{Eg} = P,$$

from which

$$v = \sqrt{\frac{Eg}{w}}. \tag{i}$$

We see that this velocity of wave propagation depends only on the modulus of elasticity E and specific weight w of the material.

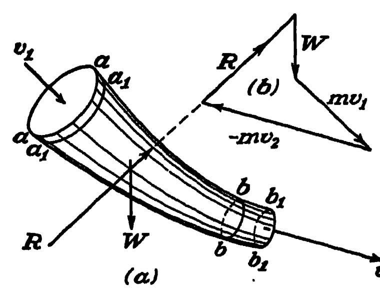


FIG. 75.

The principle of linear momentum finds a wide range of application in the field of hydrodynamics. As one example here, let us consider the case of steady flow of an incompressible fluid in a curved tube of variable cross section as shown in Fig. 75. As our system of mass particles, we consider arbitrarily that portion of the fluid contained between the cross sections aa and bb. After an infinitesimal interval of time dt, the same fluid will be contained between the cross sections a_1a_1 and b_1b_1. We

observe that the change in momentum of the fluid under consideration is due to the fact that the volume of fluid bb_1 appears in place of the volume aa_1. Denoting by Q the volume of fluid passing a given cross section in unit time and by w the specific weight, the mass rate of flow is

$$m = \frac{Qw}{g}. \tag{j}$$

Then if v_1 and v_2 are the velocities[1] of fluid at the cross sections a and b, respectively, the change in linear momentum of the fluid in time dt is represented by the vector difference

$$m \cdot dt \cdot \bar{v}_2 - m \cdot dt \cdot \bar{v}_1$$

and the rate of change of linear momentum is

$$m(\bar{v}_2 - \bar{v}_1) = \frac{Qw}{g}(\bar{v}_2 - \bar{v}_1). \tag{k}$$

The forces external to our system are: (1) the gravity force W, uniformly distributed over the volume ab, and (2) surface forces, representing pressures exerted by the walls of the tube and by adjacent fluid at the cross sections a and b. The resultant of these pressure forces we denote by R. Then the principle of linear momentum in vector form gives

$$\frac{Qw}{g}(\bar{v}_2 - \bar{v}_1) = \bar{W} + \bar{R}, \tag{81}$$

which is known as Euler's equation. From this equation, we conclude that the four vectors shown in Fig. 75b must build a closed polygon. In addition to this equation, we have from the law of continuity of flow of an incompressible fluid that

$$Q = A_1v_1 = A_2v_2, \tag{l}$$

where A and v are the cross-sectional area and velocity, respectively, at any cross section of the tube. Thus if the rate of flow of the fluid is given and the dimensions of the tube are known, we can always find the resultant force R from Eq. (81).

PROBLEMS

46. Calculate the velocity of propagation of sound in a steel bar if $E = 30(10)^6$ p.s.i. and $w = 0.284$ lb. per cu. in. *Ans.* $v = 16{,}840$ f.p.s.

47. Ten c.f.s. of water flows through a horizontal 12-in. pipe line under a pressure $p = 40$ p.s.i. as shown in Fig. 76. Calculate the resultant force R exerted on the elbow by the moving fluid. *Ans.* $R = 349$ lb.

48. A tank filled with water (specific weight w) can roll without friction on a horizontal plane as shown in Fig. 77. If a horizontal jet of cross-sectional area a

[1] We assume that the velocity distribution is uniform over any cross section.

issues from an orifice at A with velocity $v = \sqrt{2gh}$, what external force R must act on the tank at B?

Ans. $R = 2awh$.

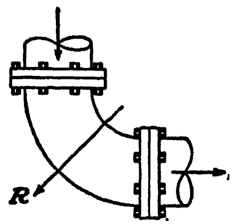


FIG. 76.

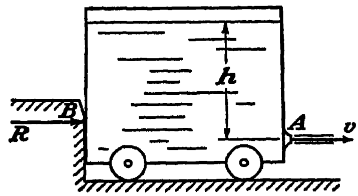


FIG. 77.

49. A tank of water has in one side a short reentrant tube of cross-sectional area A (Fig. 78). Using the principle of linear momentum, show that the cross-sectional area a of the jet at q-q must be one-half of A. Assume pressure distribution on the wall CD to be identical with that on the wall AB except for the deficit of pressure at the opening as shown in the figure.

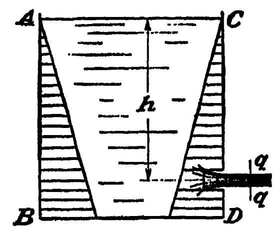


FIG. 78.

50. A water pipe of cross-sectional area A_1 empties into a larger one of cross-sectional area A_2 as shown in Fig. 79. Using the principle of linear momentum, calculate the rise in pressure between the cross sections 1 and 2 if w is the specific weight of water and v_1 and v_2 are the velocities at sections 1 and 2, respectively.

Ans. $p_2 - p_1 = (w/g)\, v_2(v_1 - v_2)$.

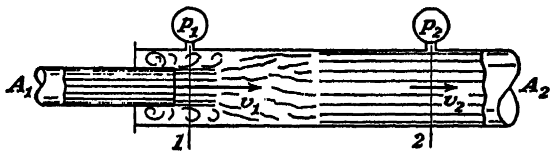


FIG. 79.

14. Rectilinear Motion of a Variable Mass: Rockets.—Occasionally we must deal with a material system the mass of which changes during motion. An engine taking water on the run or a balloon throwing away ballast as well as all jet-propelled vehicles are examples of such systems.

FIG. 80.

Using the principle of linear momentum, the differential equations of motion of such systems can be derived. Referring to Fig. 80, let us assume that the body under consideration moves rectilinearily under the action of an external force X and that during motion it continuously accumulates mass at the rate of w/g units per second.[1] For generality, we assume that the velocity of this added mass before it joins the system may be different from the velocity of the body itself. Let W/g be the

[1] If the system loses mass, we simply take w/g with negative sign.

total mass of the system at any instant of time t, and let v be its velocity. During the interval of time dt, the momentum of this mass changes by the amount $(W/g)dv$ due to change in velocity v. During the same interval of time, the additional mass $(w/g)dt$ is acquired. Denoting by v_1 the initial velocity of this additional mass in the x-direction, we see that its momentum at the instant of contact with the system changes by the amount $(v - v_1)(w/g)dt$. Thus the total change in momentum of the system in time dt is

$$\frac{W}{g}\,dv + \frac{w}{g}\,(v - v_1)dt,$$

and the principle of linear momentum gives

$$\frac{W}{g}\frac{dv}{dt} + \frac{w}{g}\,(v - v_1) = X. \tag{82}$$

This is the differential equation of rectilinear motion of a body of variable mass. As already stated, the quantity w is considered positive when the system acquires mass, otherwise negative. Likewise the velocity v_1 is considered positive when it has the same direction as v.

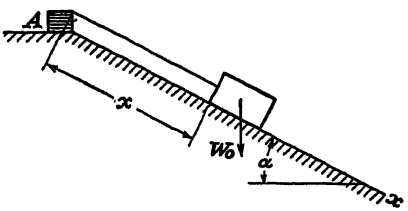


FIG. 81.

As a first application of Eq. (82), let us consider the motion along a smooth inclined plane of a weight W_0 to which the end of a chain, coiled at A, is attached (Fig. 81). Let q be the weight per unit length of this chain. Then for any displacement x of the weight W_0, the total mass in motion is

$$\frac{W}{g} = \frac{1}{g}\,(W_0 + qx). \tag{a}$$

By definition, we have also

$$w = \frac{dW}{dt} = qv. \tag{b}$$

The coiled portion of the chain being at rest, we take $v_1 = 0$ in this case. Finally, the force X is obtained as the component $(W_0 + qx)\sin\alpha$ of the gravity force, and Eq. (82) becomes

$$\frac{1}{g}\left(W\frac{dv}{dt} + v\frac{dW}{dt}\right) = W\sin\alpha$$

or

$$d(Wv) = Wg\sin\alpha\,dt \tag{c}$$

To integrate Eq. (c), we multiply both sides by Wv and substitute, on the right-hand side, $W_0 + qx$ for W and dx/dt for v. In this way, we obtain

$$Wv\,d(Wv) = (W_0 + qx)^2 g \sin\alpha\,dx. \tag{d}$$

Integrating Eq. (d), we find

$$\frac{1}{2}(Wv)^2 = \frac{g}{3q}(W_0 + qx)^3 \sin\alpha + C. \tag{83}$$

Let us assume now that when $t = 0$, the body is at rest at the top of the incline. For these initial conditions, the constant of integration C in Eq. (83) becomes

$$C = -\frac{g}{3q} W_0^3 \sin\alpha,$$

and we obtain

$$v^2 = \frac{2g}{3q}\frac{(W_0 + qx)^3 - W_0^3}{(W_0 + qx)^2}\sin\alpha. \tag{84}$$

From this equation, we can find the velocity v for any position, provided W_0 and q are known. Taking qx as small compared with W_0, we find that expression (84) reduces to

$$v^2 \approx 2gx \sin\alpha$$

as for a body of constant mass.

If a body of weight W_0 with a chain attached to it is projected vertically upward with initial velocity v_0, Eq. (83) can be used to calculate the maximum height h to which the body will ascend. In this case, we take $\alpha = -\pi/2$. Then substituting the initial conditions $v = v_0$ when $x = 0$ in Eq. (83), we find

$$C = \frac{1}{2}(W_0 v_0)^2 + \frac{g}{3q} W_0^3.$$

With this value of C, Eq. (83) becomes

$$\frac{1}{2}(Wv)^2 = \frac{g}{3q}[W_0^3 - (W_0 + qx)^3] + \frac{1}{2}(W_0 v_0)^2. \tag{85}$$

The maximum height h to which the body ascends is that value of x which makes $v = 0$ in Eq. (85). In this way, we obtain

$$(W_0 + qh)^3 = \frac{3q}{2g}(W_0 v_0)^2 + W_0^3. \tag{e}$$

Introducing the notations

$$\frac{W_0}{q} = c, \qquad \frac{v_0^2}{2g} = h_0,$$

we obtain

$$(c + h)^3 = 3c^2h_0 + c^3,$$

from which

$$h = c\sqrt[3]{1 + \frac{3h_0}{c}} - c. \tag{f}$$

If h_0 is small compared with c, the effect of the chain is negligible and, as we see from expression (f), h approaches the value h_0 in this case.

Equation (e) is valid only as long as the calculated maximum height h is less than the length of the attached chain. Otherwise we have to divide the problem into two steps. Substituting for x in Eq. (85) the length l of the chain, we calculate the velocity v_l of the projected body for this height. Starting from this height with the initial velocity v_l, the mass of the system remains constant, and it moves with constant deceleration g.

As a second application of Eq. (82), we shall discuss briefly the motion of a rocket projected vertically away from the earth. In this case the quantity w, representing the rate of discharge of gas particles, is to be considered as negative. The velocity v_1 of the gas after it leaves the rocket is also negative, and we see that the quantity $v - v_1$ represents the velocity of efflux of gas relative to the rocket. Denoting the constant value of this relative velocity by u, Eq. (82) can be written in the form

$$\frac{W}{g}\frac{dv}{dt} = X + \frac{w}{g}u. \tag{86}$$

Considering the case where the rocket is projected vertically away from the earth and neglecting air resistance, we have

$$X = -\frac{Wr^2}{(r + x)^2}, \tag{g}$$

where x is the distance from the surface of the earth of radius r. With these notations, Eq. (86) becomes

$$\frac{dv}{dt} = -\frac{gr^2}{(r + x)^2} + \frac{wu}{W}. \tag{h}$$

Let us assume now that by some device controlling the rate of efflux of gas w, the acceleration of the rocket is kept constant and take

$$\frac{dv}{dt} = \alpha g,$$

where α is a numerical factor. Observing also that

$$w = -\frac{dW}{dt}$$

and that for uniformly accelerated motion $x = \alpha g t^2/2$, we obtain, from Eq. (h),

$$-\frac{dW}{dt}\frac{u}{W} = \alpha g + \frac{gr^2}{[r + (\alpha g t^2/2)]^2}. \qquad (i)$$

Integrating Eq. (i) and noting that initially $W = W_0$, we obtain

$$-u \ln\left(\frac{W}{W_0}\right) = \alpha g t + gr^2\left[\frac{t}{2r\left(r + \frac{\alpha g t^2}{2}\right)} + \frac{1}{2r^2}\sqrt{\frac{2r}{\alpha g}}\tan^{-1}\left(\sqrt{\frac{\alpha g}{2r}}\,t\right)\right]. \qquad (87)$$

From this expression, the ratio W/W_0 for any value of t can be calculated. We know also the corresponding height $x = \alpha g t^2/2$ above the surface of the earth and the velocity $v = \alpha g t$. From these last two expressions, the relations between x and v for various values of α can be represented by curves as shown in Fig. 82.

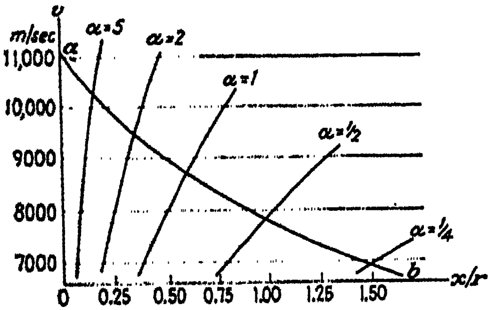


FIG. 82.

The discussed theory can now be used to examine the possibility of a rocket's reaching the moon.[1] We have to consider three stages in such motion: In the first stage, the rocket develops a high velocity and reaches a considerable height above the surface of the earth. In the second stage, it moves as a body of constant mass with deceleration; but owing to velocity acquired during the first stage, it finally reaches the neutral point where the attractions of the earth and moon balance each other. Starting from this point, the third stage of motion begins, in which the body moves with acceleration to the moon. We see that it will be possible to reach the moon if, during the first stage of motion, the rocket acquires enough velocity so that the accumulated kinetic energy will be able to carry it to the neutral point. Since during the second stage, the mass remains constant, we have to investigate the motion of a body under the action of attractive forces from both the earth and the moon. From such an analysis, we can find, for each height of the rocket above the surface of the earth, the corresponding velocity required to reach the neutral point. The results of such an analysis are represented in Fig. 82 by the curve ab. We see that as the height of the rocket increases, less and less velocity is required to bring it the rest of the way to

[1] This problem is discussed by Otto von Eberhardt, "Handbuch der physikalischen und technischen Mechanik," vol. 2, p. 268, Leipzig, 1930. See also H. Lorenz, Die Möglichkeit der Weltraumfahrt, *Z. Ver. deut. Ing.*, vol. 71, No. 19, 1927.

the neutral point. The points of intersection of the curve ab with the previously drawn α curves in Fig. 82 give, for the various values of α, the corresponding velocities v_c that the rocket must acquire during the first stage of motion. Dividing such velocities by the accelerations αg, we obtain the times t_c corresponding to the first stage. Substituting t_c for t in Eq. (87) and assuming a value for the velocity u of gas discharge, we can calculate the corresponding ratio W/W_0. This represents the fraction of the initial mass W_0/g that survives to reach the moon. In column (4) of Table XII, the values of W/W_0, assuming $u = 2{,}000$ m. per sec., are given for the

TABLE XII

(1) α	(2) v_c m./sec.	(3) x_c km.	(4) W/W_0 $u = 2{,}000$ m./sec.	(5) W/W_0 $u = 4{,}000$ m./sec.
$\frac{1}{4}$	6,800	9,440	1/66,800	1/258
$\frac{1}{2}$	7,600	6,050	1/7,685	1/87.5
1	8,650	3,800	1/1,839	1/42.7
2	9,500	2,300	1/785	1/28
5	10,200	1,060	1/431	1/20.8

values of α shown in Fig. 82. We see that even with the greatest assumed acceleration equal to $5g$, only $\frac{1}{431}$ of the initial mass of the rocket can reach the moon. Existing experiments indicate that it is difficult to get a velocity of gas discharge u much above the assumed value of 2,000 m. per sec. However, if higher velocities are obtained and $u = \beta \cdot 2{,}000$ m. per sec., where β is a numerical factor greater than unity, then the corresponding ratios W/W_0 will be obtained by raising the figures in column (4) to the power $1/\beta$. For $\beta = 2$, *i.e.*, for $u = 4{,}000$ m. per sec., the values of W/W_0 obtained in this way are shown in column (5).

From the above calculations, it is seen that even using the most favorable data, we arrive at such values for the ratio W/W_0 that a practical attainment of shooting a rocket to the moon looks doubtful under present conditions.

PROBLEMS

51. Referring to Fig. 81 and assuming that the weight ql of the entire chain is equal to the weight W_0 of the body to which it is attached, find the velocity of the system when $x = l$. Assume $v_0 = 0$ when $x = 0$. *Ans.* $v = \sqrt{7gl \sin \alpha/6}$.

52. Referring to Fig. 81 and assuming $W_0 = 0$, find the general expression for the velocity v as a function of x. Assume $v_0 = 0$ when $x = 0$.

Ans. $v = \sqrt{2gx \sin \alpha/3}$.

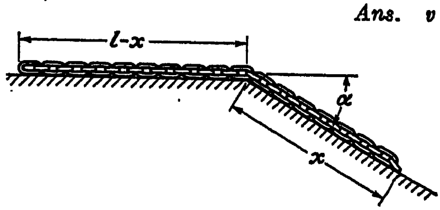


FIG. 83.

53. Repeat the solution of the preceding problem for the case where the chain is always straight as shown in Fig. 83. Assume v_0 when $x = 0$.

54. Referring to Fig. 81, assume that $\alpha = 0$ and that the body W_0 is projected with initial horizontal velocity v_0. In such case, find the velocity v as a function of displacement x. *Ans.* $v = W_0v_0/(W_0 + qx)$.

55. Two weights W_0 and W_1 are connected by a flexible but inextensible string of length l as shown in Fig. 84. If W_0 is initially projected from the plane AB with upward velocity v_0, what maximum height h will it attain? Assume that the string does not break and that its mass is negligible.

$$\textit{Ans.}\quad h = l\left[1 - \left(\frac{W_0}{W_0 + W_1}\right)^2\right] + \left(\frac{W_0}{W_0 + W_1}\right)^2 \frac{v_0^2}{2g}.$$

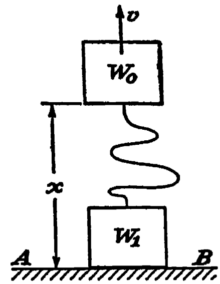


FIG. 84.

56. Denoting the mass of the earth by m and its radius by r and taking the mass of the moon as $m/75$ and the distance between centers of the two bodies as $60r$, determine the distance x from the center of the earth to the neutral point where the attractions of the earth and moon on a given body will just balance each other. *Ans.* $x = 53.79r$.

15. Principle of Angular Momentum.—Referring to Fig. 85*a*, let us consider a particle A of mass m moving with velocity v. Its momentum is represented by a vector mv having the direction of the velocity. Multiplying this momentum vector by its distance OD from an arbitrarily chosen fixed point O, we obtain the *moment of momentum* of the particle with respect to that point. This moment of momentum, equal to the doubled area of the ΔOAB, can be represented by a vector H' perpendicular to the plane OAB and directed in accordance with the right-hand screw rule as shown. In the case of a system of particles, the geometric sum of such individual moments of momentum is called the *angular momentum* of the system and will be denoted by H.

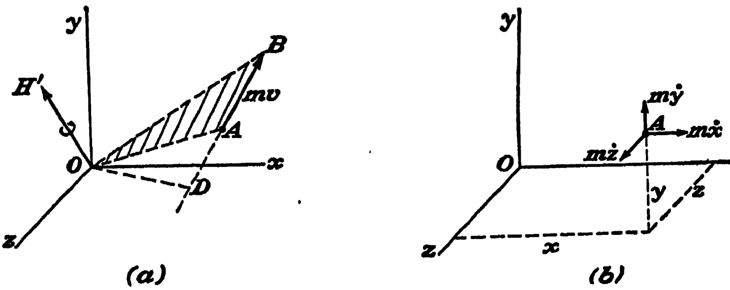


FIG. 85.

Resolving the momentum $\overline{mv}$ of the particle A in Fig. 85*a* into orthogonal components $m\dot{x}$, $m\dot{y}$, and $m\dot{z}$ (Fig. 85*b*), we can represent the moment of momentum H' by three components. Since the moment of the resultant momentum is equal to the algebraic sum of the corresponding moments of its components, we conclude that the projection of the vector H' on, say, the x-axis is

$$\left.\begin{aligned} H_x' &= m(\dot{z}y - \dot{y}z), \\ &\ldots\ldots\ldots\ldots\ldots, \\ &\ldots\ldots\ldots\ldots\ldots \end{aligned}\right\} \qquad (a)$$

Similar expressions for H_y' and H_z', as indicated by dots, can be written. Summing expressions (*a*) for all particles of the system, we obtain, for the projections of the angular momentum of the system,

$$\left.\begin{aligned} H_x &= \Sigma[m(\dot{z}y - \dot{y}z)], \\ &\ldots\ldots\ldots\ldots\ldots\ldots, \\ &\ldots\ldots\ldots\ldots\ldots\ldots \end{aligned}\right\} \qquad (b)$$

To illustrate the calculation of the angular momentum of a system of particles in a given case, let us consider a rigid body rotating about a fixed axis OA with angular velocity $\dot{\theta}$ (Fig. 86). Taking coordinate axes x, y, z through point O and letting z coincide with the axis of rotation, we see that a particle of mass m at the distance r from this axis describes a circular path parallel to the xy-plane and the components of its velocity $r\dot{\theta}$ are

$$\dot{x} = -y\dot{\theta}, \qquad \dot{y} = +x\dot{\theta}, \qquad \dot{z} = 0.$$

Substituting these expressions into Eqs. (*b*) and observing that $\dot{\theta}$ is the same for all particles, we obtain

$$\left.\begin{aligned} H_x &= -\dot{\theta}\,\Sigma mxz = -I_{xz}\dot{\theta}, \\ H_y &= -\dot{\theta}\,\Sigma myz = -I_{yz}\dot{\theta}, \\ H_z &= \dot{\theta}\,\Sigma mr^2 = I_z\dot{\theta}, \end{aligned}\right\} \qquad (c)$$

where I_{xz} and I_{yz} are the products of inertia of the body with respect to the axes xz and yz, respectively, and I_z is its moment of inertia with respect to the axis of rotation. In the particular case where the axis of rotation is one of three principal axes through point O, the products of inertia I_{xz} and I_{yz} vanish and the resultant angular momentum of the rotating body coincides with the z-axis and has the magnitude $I_z\dot{\theta}$.

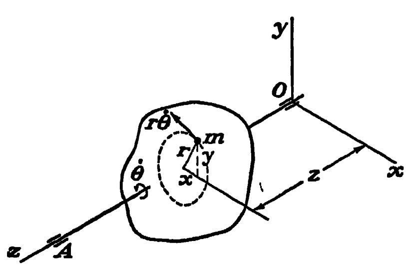


FIG. 86.

We proceed now to establish the relation between the angular momentum of a system of particles and the forces acting thereon. For this purpose, we begin with the equations of motion for a single particle as represented by Eqs. (*a*) on page 107. Multiplying the second of these equations by z and the third by y and subtracting one from the other, we obtain

$$\left.\begin{array}{l}\dfrac{d}{dt}[m(\dot{z}y - \dot{y}z)] = (Z_e + Z_i)y - (Y_e + Y_i)z, \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots, \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\end{array}\right\} \tag{d}$$

As can be seen by comparison with the first of expressions (a) above, the left-hand side of this equation represents the rate of change of the moment of momentum of the particle with respect to the x-axis. Likewise, the right-hand side of the equation represents the moment of all forces acting on the particle with respect to the same axis. Thus the equation states that the rate of change of moment of momentum of the particle with respect to the fixed x-axis is equal to the corresponding moment of all forces acting on the particle. Similar equations, as indicated by dots, can be obtained for the y- and z-axes. Writing Eqs. (d) for each particle of a system, summing them up, and observing that the internal forces, represented by X_i, Y_i, Z_i, always occur in balanced pairs and cancel each other during summation, we obtain

$$\left.\begin{array}{l}\dfrac{d}{dt}\sum m(\dot{z}y - \dot{y}z) = \sum (Z_e y - Y_e z), \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots, \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\end{array}\right\} \tag{88}$$

Using notations (b) and introducing similar notations

$$\left.\begin{array}{l}M_x = \Sigma(Z_e y - Y_e z), \\ \dots\dots\dots\dots\dots\dots, \\ \dots\dots\dots\dots\dots\dots\end{array}\right\} \tag{e}$$

for the moments of external forces with respect to the coordinate axes, Eqs. (88) can be written in the following more compact form:

$$\frac{d}{dt}(H_x) = M_x, \qquad \frac{d}{dt}(H_y) = M_y, \qquad \frac{d}{dt}(H_z) = M_z. \tag{89}$$

Since the coordinate axes x, y, z were chosen arbitrarily, Eqs. (89) may be said to state that the rate of change of angular momentum of a system of particles with respect to any fixed axis in space is equal to the resolved moment of external forces with respect to the same axis. This statement represents the principle of angular momentum.

If the angular momentum H of a system of particles is represented vectorially, the rate of change of this vector with respect to time will be represented by the velocity of its end point. On the basis of Eqs. (89), we can state that the velocity of the end of the angular momentum vector with respect to a fixed point O represents the resultant moment, with

respect to the same point, of all external forces acting on the system. In the particular case where the moment of external forces with respect to a fixed point vanishes, we conclude that the angular momentum vector H remains constant in magnitude and direction.

Sometimes, in calculating the angular momentum of a system of particles, it is advantageous to use the notion of mass-center having coordinates x_c, y_c, z_c referred to fixed axes x, y, z. Taking through this point a system of coordinate axes ξ, η, ζ parallel, respectively, to x, y, z and moving with the mass-center, the coordinates of any particle with respect to the fixed axes will be

$$x = x_c + \xi, \qquad y = y_c + \eta, \qquad z = z_c + \zeta, \tag{f}$$

from which, by differentiation with respect to time,

$$\dot{x} = \dot{x}_c + \dot{\xi}, \qquad \dot{y} = \dot{y}_c + \dot{\eta}, \qquad \dot{z} = \dot{z}_c + \dot{\zeta}. \tag{g}$$

Substituting expressions (f) and (g) into Eqs. (b) above, we obtain

$$\left.\begin{aligned} H_x = (\dot{z}_c y_c - \dot{y}_c z_c)\Sigma m + \Sigma m(\dot{\zeta}\eta - \dot{\eta}\zeta) \\ + \dot{z}_c\, \Sigma m\eta + y_c\, \Sigma m\dot{\zeta} - \dot{y}_c\, \Sigma m\zeta - z_c\, \Sigma m\dot{\eta}, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \end{aligned}\right\} \tag{h}$$

Since the origin of the ξ, η, ζ coordinates always coincides with the mass-center of the system, we have, by virtue of Eqs. (b) page 107,

$$\Sigma m\eta = \Sigma m\dot{\zeta} = \Sigma m\zeta = \Sigma m\dot{\eta} = 0$$

and expressions (h) reduce to

$$\left.\begin{aligned} H_x = (\dot{z}_c y_c - \dot{y}_c z_c)\Sigma m + \Sigma m(\dot{\zeta}\eta - \dot{\eta}\zeta), \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \end{aligned}\right\} \tag{i}$$

From these expressions, we see that the calculation of the angular momentum of a system of particles can be made in two steps: First, we assume that the entire mass M of the system is concentrated at the mass-center and moving with the velocity of that point. This gives us that part of the angular momentum represented by the first term in each of expressions (i). The second term is obtained by considering only the relative motions of the particles with respect to the moving axes ξ, η, ζ and then calculating the angular momentum with respect to these axes as if they were immovable.

Using expressions (i) for angular momentum together with expressions (f) in Eqs. (88) representing the principle of angular momentum, we obtain

$$\left.\begin{aligned} \frac{d}{dt}M(\dot{z}_c y_c - \dot{y}_c z_c) + \frac{d}{dt}\sum m(\dot{\zeta}\eta - \dot{\eta}\zeta) \qquad\qquad & \\ = \sum Z_e y_c - \sum Y_e z_c + \sum Z_e \eta - \sum Y_e \zeta, & \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots, & \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots & \end{aligned}\right\} \quad (j)$$

These equations can be greatly simplified by using Eqs. (79) of Art. 13 (see page 108). For example, multiplying the second equation there by z_c and the third by y_c and subtracting one from the other, we obtain

$$\left.\begin{aligned} \frac{d}{dt}M(\dot{z}_c y_c - \dot{y}_c z_c) = \sum Z_e y_c - \sum Y_e z_c, & \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots, & \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots & \end{aligned}\right\} \quad (k)$$

Similar expressions, as indicated by dots, can be obtained in the same way from the first and third and the first and second of Eqs. (79). In virtue of Eqs. (k), Eqs. (j) reduce to

$$\left.\begin{aligned} \frac{d}{dt}\sum m(\dot{\zeta}\eta - \dot{\eta}\zeta) = \sum Z_e \eta - \sum Y_e \zeta, & \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots, & \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots & \end{aligned}\right\} \quad (90)$$

These equations have the same form as Eqs. (88), which indicates that the principle of angular momentum, previously formulated for fixed axes, holds also for axes moving with the mass-center of the system. At any instant and for any axis passing through the instantaneous position of the mass-center, the rate of change of angular momentum is equal to the moment of all external forces with respect to that axis. In calculating the angular momentum of the system with respect to such an axis, it makes no difference whether we use the actual momenta of the various particles or consider only the momenta of relative motion. This follows from the identity of expressions (h) and (i) above.

We have already noted from Eqs. (89) that if the resultant moment of external forces with respect to a fixed point O vanishes during motion of the system, then the projections of the angular momentum on fixed coordinate axes through O remain constant. Now in the same manner, we conclude from Eqs. (90) that if the moment of external forces with respect to the mass-center of the system vanishes, then the angular momentum of the system with respect to its own mass-center remains constant. Taking, for example, the solar system and neglecting the action thereon of fixed stars, we conclude that the angular momentum

of the system with respect to its mass-center is invariable. Since the orbits of the planets are known, the magnitude and direction of this angular momentum can be calculated. The plane normal to the angular momentum vector always retains the same orientation in space and is called the *invariable plane.* Laplace[1] suggested that it be used as a reference plane in astronomy.

By way of application of the principle of angular momentum, let us consider the device shown in Fig. 87. A horizontal bar, carrying two identical and symmetrically placed masses m, is pivoted inside a heavy frame AA and connected thereto by a helical spring so that it can perform torsional oscillations about the vertical axis OO. The frame AA, in turn, is suspended by a fine thread and can also turn freely about the axis OO. Suppose now that the bar mm is rotated by an angle θ_0 with respect to the frame AA and then the system is released from rest; torsional oscillations of both the bar and the frame will ensue. However, since the moment of external forces with respect to the vertical axis OO is zero, the angular momentum of the system cannot change; and since it was zero at the instant of release, it must remain zero. From this we conclude that if I_1 and I_2 denote, respectively, the moments of inertia of the frame and bar about the axis OO and $\dot\theta_1$ and $\dot\theta_2$ their angular velocities, then we must have

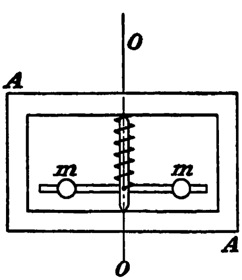


FIG. 87.

$$I_1\dot\theta_1 + I_2\dot\theta_2 = 0,$$

from which

$$\frac{\dot\theta_1}{\dot\theta_2} = -\frac{I_2}{I_1}, \tag{l}$$

We see that the angular velocities are always in opposite directions and inversely proportional to the moments of inertia. Since the ratio of angular velocities is constant, it follows that the amplitudes of oscillation of the two parts of the system are in the same ratio as the angular velocities. Neglecting friction, we conclude also that when the two parts of the system occupy extreme positions, the angle of twist in the spring is θ_0. Hence the amplitudes of oscillation are

$$(\theta_1)_{\max} = \frac{I_2\theta_0}{I_1 + I_2}, \qquad (\theta_2)_{\max} = \frac{I_1\theta_0}{I_1 + I_2}. \tag{m}$$

[1] Laplace treated the planets as particles and neglected the angular momentum due to rotation about their own axes. The necessary corrections were made later by Poinsot. See appendix to his famous book, "Elements de statique," 8th ed., Paris, 1842, p. 343.

The two extreme positions of the system are shown in Fig. 88. Taking Fig. 88*a* as the initial position, we see that the middle plane, denoted by the line *aa*, is displaced with respect to the initial position of the frame by the positive angle $(\theta_1)_{max}$, the magnitude of which increases with an increase of the moment of inertia I_2 of the bar *mm*. If, during motion of the system, we move the masses *m* away from the axis of rotation at the instant when the bar is in the extreme position *mm* (Fig. 88*a*), the angle $(\theta_1)_{max}$ increases and the middle plane *aa* will be rotated slightly in the positive direction as indicated by an arrow. Likewise by decreasing I_2 in this position, the middle plane *aa* will be shifted slightly in the negative direction. On the other hand, if the moment of inertia I_2 is increased when the bar is in the other extreme position m_1m_1 (Fig. 88*b*), the middle plane *aa* will be shifted slightly in the negative direction and vice versa. From these observations, it follows that if, during motion of the system, we always move the masses *m* outward in the one extreme position (Fig. 88*a*) and inward in the other extreme position (Fig. 88*b*), the middle plane *aa* will rotate consistently in the positive direction. Thus, without the aid of external forces, the system as a whole can change its orientation in space.[1]

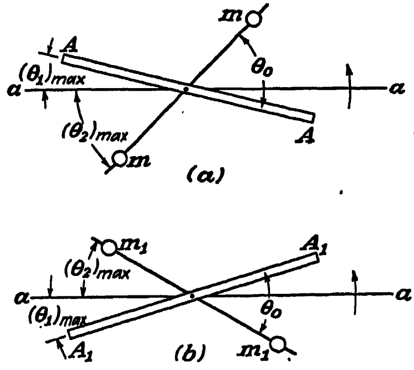


FIG. 88.

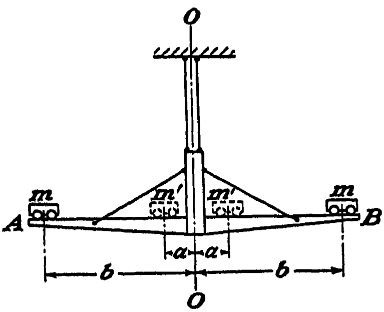


FIG. 89.

To prove rotation of the earth around its polar axis, the device shown in Fig. 89 can be used.[2] A long beam *AB* with bifilar suspension carries two symmetrically placed masses *m* as shown. Initially the system is at rest with respect to the earth, and the masses *m* are at the extremities of the beam. In such condition, the sys-

[1] This explains how a cat dropped feet up from a window can land on its feet. See paper by M. Marey, *Compt. rend.*, vol. 119, 1894. Reproduced also in *Nature*, Nov. 22, 1894.

[2] This was first suggested by L. Poinsot, *Compt. rend.*, vol. 32, p. 206, 1851. The experiments were actually made by J. G. Hagen, *Z. Instrumentenk.*, vol. 40, p. 65, 1920.

tem has, with respect to the vertical axis OO, the angular momentum

$$H_z = (I_0 + 2mb^2)\omega_0 \sin \phi,$$

where I_0 is the moment of inertia of the beam AB about the axis OO, ω_0 is the angular velocity of the earth about its polar axis, and ϕ is the latitude of the site of the experiment. Since the moment of external forces (gravity) about the axis OO vanishes, it follows from the principle of angular momentum that if the masses m are moved inward to $m'm'$, the new angular momentum of the system must be equal to the old and we write

$$(I_0 + 2mb^2)\omega_0 \sin \phi = (I_0 + 2ma^2)(\omega_0 \sin \phi + \omega'),$$

where ω' denotes the acquired angular velocity of the system relative to the earth. From this expression, we obtain

$$\omega' = \frac{2m(b^2 - a^2)\omega_0 \sin \phi}{I_0 + 2ma^2}. \qquad (n)$$

We see that if the earth has angular velocity about its polar axis, the system will be brought into relative rotation with respect to the earth simply as a result of changing the radial distances of the masses m. Needless to say, the experiment will require great finesse.

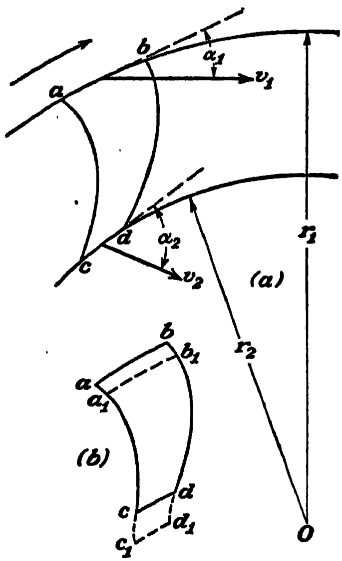


FIG. 90.

As with the principle of linear momentum, the principle of angular momentum finds a wide application in the field of hydrodynamics. As an example, let us consider the wheel of a reaction turbine with vertical axis as shown in Fig. 90. Water under pressure flows from the outside to the inside of this wheel through such channels as the one shown. We denote by v_1 the absolute velocity of the water entering the channel and by α_1 the angle that it makes with the tangent to the outer circumference of the wheel. Likewise, v_2 denotes the absolute velocity of water leaving the channel, and α_2 the angle that it makes with the tangent to the inner circumference of the wheel. As our system of mass particles, we take the water that at any instant t is contained between the cross sections ab and cd of one channel (Fig. 90b). After an infinitesimal interval of time dt, the same water will occupy the volume $a_1b_1c_1d_1$. Assuming a steady flow, we may now calculate the change in angular momentum of this

water with respect to the vertical axis OO of the wheel. With this assumption, the velocities of the particles of water in the space a_1b_1cd remain unchanged during the interval dt and we have only to observe that the particles which at the beginning of the interval were in the space aba_1b_1 are replaced at the end of the interval by particles in the space cdc_1d_1. The required change in angular momentum therefore is obtained by subtracting the angular momentum of the mass aba_1b_1 from that of the mass cdc_1d_1. Let q denote the rate of flow (cubic feet per second) through one channel, w the specific weight of water, and r_1 and r_2 the outer and inner radii, respectively, of the wheel. Then the discussed change in angular momentum is

$$\frac{qw}{g}(v_2\cos\alpha_2r_2 - v_1\cos\alpha_1r_1)dt.$$

Considering now the entire wheel and denoting the total rate of flow by Q, we find that the rate of change of angular momentum is

$$\frac{Qw}{g}(v_2\cos\alpha_2r_2 - v_1\cos\alpha_1r_1). \tag{o}$$

From the principle of angular momentum, we conclude now that this is equal to the moment, with respect to the axis of rotation, of all external forces exerted on the water as it passes through the channels. The moment of gravity forces vanishes in virtue of the verticality of the axis OO. The moments of pressure forces over the surfaces ab and cd also vanish, since these forces are in the radial direction. Hence, we conclude that expression (o) represents the moment of reactions of the wheel on the flowing water. Changing the sign of expression (o), we obtain the pressure moment of the water on the wheel. This quantity represents the torque developed by the turbine, and we write[1]

$$M_t = \frac{Qw}{g}(v_1\cos\alpha_1 r_1 - v_2\cos\alpha_2 r_2). \tag{91}$$

Multiplying M_t by the angular velocity ω of the wheel, we obtain the power developed by the turbine. Thus when $M_t\omega$ is in *foot-pound-second* units

$$\text{hp.} = \frac{M_t\omega}{550}. \tag{p}$$

PROBLEMS

57. Assuming that icecaps of total mass m melt from the polar regions of the earth of mass M, radius r, and radius of gyration i_c and that the water spreads evenly over

[1] This equation is commonly known as Euler's turbine formula.

the entire surface of the globe, calculate the change $\delta\omega$ in the angular velocity ω of the earth about its axis. *Ans.* $\delta\omega = -2m\omega r^2/3Mi_e^2$.

58. Calculate the numerical value of $\delta\omega$ in the preceding problem if the mass of ice m is sufficient to form a layer of water of depth $d = 0.066$ ft. over the earth. Assume the earth to be a homogeneous sphere of radius $r = 20.9(10)^6$ ft. and specific gravity $s = 5.25$.

Ans. $-\delta\omega = 0°\ 00'\ 01.''$ per year.

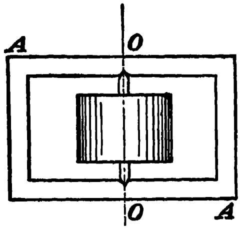


FIG. 91.

59. A frame AA having moment of inertia I_1 about the vertical axis OO is suspended by a fine thread and contains a rotor of moment of inertia I_2 as shown in Fig. 91. Initially, the rotor has an angular velocity ω_0, and the frame is at rest. Due to friction, the frame is brought into rotation also (the thread is assumed to offer no resistance to twist). Compute the friction torque (assumed constant) if t sec. of time are required for the rotor to come to rest with respect to the frame. *Ans.* $M_f = \dfrac{I_1I_2\omega_0}{(I_1 + I_2)t}$.

60. A circular turntable can rotate freely about its vertical geometric axis OO and has moment of inertia I_0 with respect thereto. On this turntable, a toy locomotive of mass m runs on a circular track of radius r and with O as a center. Initially, turntable and engine are both at rest. What angular velocity ω will the turntable acquire if the engine begins to move and acquires a velocity v relative to its track?

Ans. $\omega = mrv/(I_0 + mr^2)$.

61. A uniform circular wire of radius r and mass m rests on a perfectly smooth horizontal plane and is constrained to rotate about a point O on its circumference. A particle of mass m, initially at a point A diametrically opposite the fixed point O, begins to move with constant velocity v along the wire. Prove that after a lapse of time t, the diameter OA will have rotated through the angle

$$\theta = \frac{vt}{2r} - \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{1}{\sqrt{3}}\tan\frac{vt}{2r}\right).$$

62. A particle of mass m moves along a straight slender bar of length $2a$ and mass m. The extremities of this bar are constrained to remain on a fixed horizontal circle of radius $r = 2a/\sqrt{3}$. Initially, the particle occupies the mid-point of the bar, and both are at rest. If from this condition, the particle begins to move with velocity v along the bar, show that after a lapse of time t, the bar will have rotated through the angle

$$\theta = \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{vt}{a}\right).$$

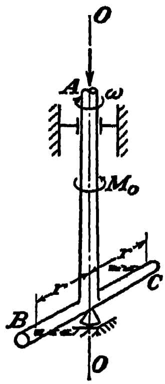


FIG. 92.

63. Figure 92 represents a so-called *Segner's turbine.* Water flows into the vertical tube at A and out through the orifices at B and C. If the system rotates with angular velocity ω about the vertical axis OO, find the torque M_0 resisting rotation. The rate of flow in cubic feet per second is Q, and the velocity of efflux relative to the arm BC is v. *Ans.* $M_0 = (Qw/g)(v - r\omega)r$.

64. The horizontal wheel of a reaction turbine has outer radius $r_1 = 3$ ft., inner radius $r_2 = 2$ ft., and height $h = 1$ ft. as shown in Fig. 93. The angles of entrance and exit (see Fig. 90) are, respectively, $\alpha_1 = 30$ deg. and $\alpha_2 = 60$ deg. The total rate of flow $Q = 100$ c.f.s. Compute the torque M_t from Eq. (91).

Ans. $M_t = 1{,}127$ ft.-lb.

16. Impact.—Let us suppose that a system of particles moving in some prescribed manner collides suddenly with an obstruction; such collision is called an *impact*. During this impact, which is of very short duration, extremely large forces act momentarily on the system at the points of impact and produce definite changes in the velocities of various particles of the system. These changes in velocity take place almost instantaneously, so that the system finds itself, at the end of the impact, in essentially the same configuration that it had at the beginning but with a completely new motion. It is desired to find the new motion of the system after impact. In dealing with this problem, we shall be justified to neglect, during impact, all ordinary forces such as gravity, etc., in comparison with the very large forces of impact and also to neglect any change in configuration of the system during the impact. Thus the problem is reduced to finding the instantaneous changes in the velocities of the particles as a result of the impact alone.

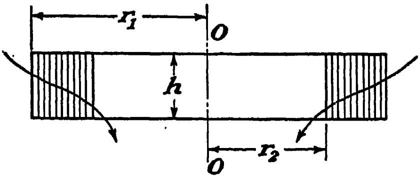


FIG. 93.

The principles of linear and angular momentum discussed in Arts. 13 and 15 can be used to advantage in dealing with the above problem of impact. We begin with Eqs. (77), page 107, representing the principle of linear momentum. Multiplying each of these equations by dt and integrating from t_0, the beginning of impact, to t_1, the end of impact, we obtain

$$\left.\begin{array}{l}\sum m(\dot{x}' - \dot{x}) = \sum \int_{t_0}^{t_1} X_e dt, \\ \dots\dots\dots\dots\dots\dots, \\ \dots\dots\dots\dots\dots\dots,\end{array}\right\} \qquad (a)$$

where $\dot{x}$, $\dot{y}$, $\dot{z}$ are the components of the velocity of a particle just before impact and $\dot{x}'$, $\dot{y}'$, $\dot{z}'$ are the corresponding components just after impact. Treating Eqs. (88), page 121, representing the principle of angular momentum, in the same way, we obtain

$$\left.\begin{array}{l}\sum m[y(\dot{z}' - \dot{z}) - z(\dot{y}' - \dot{y})] = \sum \left(y \int_{t_0}^{t_1} Z_e dt - z \int_{t_0}^{t_1} Y_e dt\right), \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots, \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots,\end{array}\right\} \qquad (b)$$

where the coordinates x, y, z of any particle are treated as constants during the interval of impact.

Equations (a) state that during impact the change in linear momentum of the system in any direction is equal to the total impulse of external

impact forces in the same direction. Likewise, Eqs. (b) state that the change in angular momentum of the system about any fixed axis is equal to the summation of moments of external impulses with respect to that axis. Equations (b), of course, hold also for coordinate axes moving with the mass-center of the system. Using the symbol Δ to denote the variation, during impact, of linear and angular momenta and introducing for impulses the notations

$$X_e' = \int_{t_0}^{t_1} X_e dt, \qquad Y_e' = \int_{t_0}^{t_1} Y_e dt, \qquad Z_e' = \int_{t_0}^{t_1} Z_e dt, \tag{c}$$

Eqs. (a) and (b) can be presented more compactly as follows:

$$\Delta(\Sigma m\dot{x}) = \Sigma X_e'; \qquad \Delta[\Sigma m(\dot{z}y - \dot{y}z)] = \Sigma(Z_e'y - Y_e'z); \tag{92}$$
$$\Delta(\Sigma m\dot{y}) = \Sigma Y_e'; \qquad \Delta[\Sigma m(\dot{x}z - \dot{z}x)] = \Sigma(X_e'z - Z_e'x); \tag{93}$$
$$\Delta(\Sigma m\dot{z}) = \Sigma Z_e'; \qquad \Delta[\Sigma m(\dot{y}x - \dot{x}y)] = \Sigma(Y_e'x - X_e'y). \tag{94}$$

In the following paragraphs, we shall consider the application of these equations to several particular cases of impact and see how they may be used to determine the motion of a system after an impact if the motion before the impact was known.

As a first example, we consider a thin plate, with center of gravity at C, which initially has a prescribed motion in its own plane (Fig. 94). Assume now that a certain point O of the plate, at the distance r from C, is suddenly fixed so that subsequent motion must consist of a rotation about an axis through O and normal to the plane of the plate. What will be the angular velocity ω' of this rotation about O? To answer this question, we choose fixed coordinate axes x, y, z through point O as shown, the x-axis coinciding with the position of the line OC at the instant of impact and the z-axis normal to the plane of the plate. Due to the sudden fixing of point O, there will be an impact at this point, but the moment of the corresponding impulse with respect to the z-axis vanishes. Thus the second of Eqs. (94) states that during impact, the angular momentum of the system with respect to the z-axis remains constant. The motion of the plate before impact can be defined by the components $\dot{x}_c$ and $\dot{y}_c$ of the velocity of its center of gravity C and by the angular velocity ω with respect to that center. Then, just before impact, the angular momentum of the system with respect to the fixed z-axis was

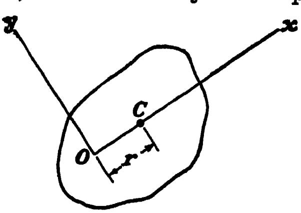


FIG. 94.

$$H_z = M(\dot{y}_c r + i_c^2\omega), \tag{d}$$

where M is the total mass of the plate and i_c is its radius of gyration with

respect to the axis through C and parallel to z. Likewise, after impact, the angular momentum with respect to the z-axis is

$$H_z' = M(i_c^2 + r^2)\omega'. \tag{e}$$

Equating expressions (d) and (e), we obtain

$$\omega' = \frac{\dot{y}_c r + i_c^2\omega}{i_c^2 + r^2}. \tag{f}$$

Having the angular velocity ω' of the plate after impact, we can easily calculate the impulse that occurred at O. From the first each of Eqs. (92) and (93), we have, for the components of this impulse,

$$X' = -M\,\Delta\dot{x}_c, \qquad Y' = M\,\Delta\dot{y}_c. \tag{g}$$

The changes $\Delta\dot{x}_c$ and $\Delta\dot{y}_c$ in the velocity of the center of gravity are

$$\Delta\dot{x}_c = 0 - \dot{x}_c \qquad \text{and} \qquad \Delta\dot{y}_c = r\omega' - \dot{y}_c = \frac{i_c^2}{i_c^2 + r^2}(r\omega - \dot{y}_c).$$

Using these values in Eqs. (g), we find

$$X' = -M\dot{x}_c, \qquad Y' = \frac{Mi_c^2}{i_c^2 + r^2}(r\omega - \dot{y}_c). \tag{h}$$

As a second example, we consider a plate that initially rotates with angular velocity ω about an instantaneous axis Ox in its own plane (Fig. 95). At a certain instant t_0, another axis Ox_1, also in the plane of the plate but inclined to Ox by an angle α, is suddenly fixed so that subsequent rotation of the system must be around this axis. It is required to find the new angular velocity around Ox_1. We see that corresponding to the initial motion, any mass particle of the system has the velocity $y\omega$ normal to the xy-plane. The corresponding initial angular momentum of the system with respect to Ox_1 is

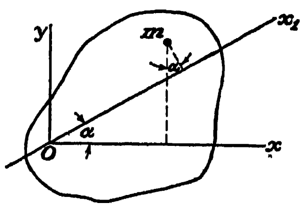


FIG. 95.

$$H_1 = \int_A \frac{w}{g}\,dA\omega y(y\cos\alpha - x\sin\alpha), \tag{i}$$

where w is the weight per unit area of the plate and the integration is extended over the total area A. After the impact, when the plate is rotating about the fixed axis Ox_1 with angular velocity ω', its angular momentum with respect to this axis is

$$H_1' = \int_A \frac{w}{g}\,dA\omega'(y\cos\alpha - x\sin\alpha).^2 \tag{j}$$

Since the impulsive forces acting on the plate during impact are distributed along Ox_1, their moments with respect to this axis vanish and we conclude accordingly that expressions (i) and (j) are equal. From this condition, we obtain

$$\omega' = \omega \frac{I_x \cos \alpha - I_{xy} \sin \alpha}{I_x \cos^2 \alpha - I_{xy} \sin 2\alpha + I_y \sin^2 \alpha}, \tag{k}$$

where I_x, I_y, and I_{xy} are the moments of inertia and product of inertia of the plate with respect to the x- and y-axes. We see that if the axis Ox_1 is at right angles to Ox, expression (k) for ω' reduces to

$$\omega' = -\omega \frac{I_{xy}}{I_y}. \tag{k'}$$

Also in the particular case where x and y are principal axes of the plate, we have

$$\omega' = \omega \frac{I_x \cos \alpha}{I_x \cos^2 \alpha + I_y \sin^2 \alpha}. \tag{k''}$$

In this case, if $\alpha = 90$ deg., the plate is brought to rest by the impact.

As a third example, let us consider the impact of a ball of radius a falling onto a rough horizontal plane (Fig. 96).[1] The motion of the ball just before impact is defined by the velocity v_c of its center, directed as shown, and by its angular velocity ω. After impact, the motion will be defined by the velocity components $\dot{x}_c'$, $\dot{y}_c'$ and the angular velocity ω'. To find these quantities defining the new motion, we use the first of Eqs. (92) and (93) and the last of Eqs. (94). Denoting by X' and Y' the components of the impulse at the point of contact O and by M the mass of the ball, these equations become

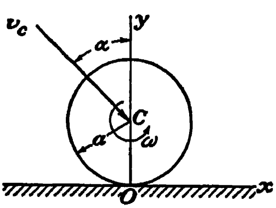


FIG. 96.

$$\left.\begin{aligned} M(\dot{x}_c' - v_c \sin \alpha) &= X', \\ M(\dot{y}_c' + v_c \cos \alpha) &= Y', \\ \tfrac{2}{5} M a^2 (\omega' - \omega) &= X'a. \end{aligned}\right\} \tag{l}$$

The last of these equations has been written with respect to a moving z-axis coinciding with the center of gravity of the ball.

We now make the assumption that the impact is completely inelastic and that there is sufficient friction to prevent slipping at the point of contact O. On the basis of these assumptions, we have

$$\dot{y}_c' = 0, \qquad \omega' a + \dot{x}_c' = 0. \tag{m}$$

[1] This discussion will apply to the motion of a ball projected onto a bowling alley.

Using these conditions in Eqs. (l) and solving for X', Y', we find

$$\left.\begin{aligned} X' &= -\tfrac{2}{7}M(\omega a + v_c \sin\alpha), \\ Y' &= Mv_c \cos\alpha \end{aligned}\right\} \qquad (n)$$

defining the components of the impulse and

$$\left.\begin{aligned} \omega' &= \frac{2}{7}\omega - \frac{5}{7}\frac{v_c}{a}\sin\alpha, \\ \dot{x}_c' &= -\omega a = \tfrac{5}{7}v_c \sin\alpha - \tfrac{2}{7}a\omega \end{aligned}\right\} \qquad (o)$$

defining motion of the ball after impact.

The results (n) and (o) depend on the assumption that there is no sliding at the point of contact O. This requires that the coefficient of friction μ satisfy the condition

$$\mu > \left|\frac{X'}{Y'}\right|.$$

Substituting the obtained values (n) for X' and Y', this becomes

$$\mu > \frac{2}{7}\left(\frac{a\omega}{v_c \cos\alpha} + \tan\alpha\right). \qquad (p)$$

For

$$\mu < \frac{2}{7}\left(\frac{a\omega}{v_c \cos\alpha} + \tan\alpha\right), \qquad (q)$$

there will be slipping at the point of contact O and we have

$$X' = -\mu Y' = -\mu M v_c \cos\alpha. \qquad (r)$$

Using this value of X' in the first and last of Eqs. (l), we find

$$\left.\begin{aligned} \dot{x}_c' &= v_c(\sin\alpha - \mu\cos\alpha), \\ \omega' &= \omega - \frac{5}{2}\mu\frac{v_c}{a}\cos\alpha. \end{aligned}\right\} \qquad (s)$$

If, instead of a completely inelastic impact, we have a coefficient of restitution e,* we must take, instead of $\dot{y}_c' = 0$,

$$\dot{y}_c' = ev_c \cos\alpha.$$

Then the second of Eqs. (l) gives

$$Y' = M(1 + e)v_c \cos\alpha.$$

If we assume again that there is no sliding during contact with the plane, Eqs. (o) are still valid in this case and we obtain

$$\frac{\dot{x}_c'}{\dot{y}_c'} = \frac{1}{e}\left(\frac{5}{7}\tan\alpha - \frac{2}{7}\frac{a\omega}{v_c \cos\alpha}\right). \qquad (t)$$

* See the authors' "Engineering Mechanics," 2d ed., p. 338.

We see from this expression that it is possible to have such a relation between the initial velocities ω and v_o that $\dot{x}_o'/\dot{y}_o' = -\tan\alpha$, in which case the ball, after striking the plane, will rebound in the direction opposite to its initial motion. Again, the assumption of no slipping requires, for the coefficient of friction,

$$\mu > \left|\frac{X'}{Y'}\right| = \frac{\frac{2}{7}}{1+e}\left(\frac{a\omega}{v_o\cos\alpha} + \tan\alpha\right). \tag{u}$$

For

$$\mu < \frac{\frac{2}{7}}{1+e}\left(\frac{a\omega}{v_o\cos\alpha} + \tan\alpha\right). \tag{v}$$

there will be sliding at the point of contact O and, as before, we obtain

$$X' = -\mu Y' = -\mu(1+e)Mv_o\cos\alpha. \tag{w}$$

The first and last of Eqs. (l) then give

$$\left.\begin{aligned} \dot{x}_o' &= v_o[\sin\alpha - \mu(1+e)\cos\alpha], \\ \omega' &= \omega - \frac{5}{2}\mu(1+e)\frac{v_o}{a}\cos\alpha. \end{aligned}\right\} \tag{x}$$

As a last example, let us consider a thin plate bounded by the arc OB of a parabola, the axis Ox, and an ordinate BA (Fig. 97). The vertex O of the parabola is a fixed point, and initially the plate is at rest in a horizontal position. It is desired to find the motion of the plate immediately following a vertical impulse Z' representing a sharp blow applied at B. Such a blow produces rotation of the plate about some horizontal axis passing through the fixed point O. Let ω_x and ω_y denote the components of the angular velocity about this unknown axis. Then the velocity of any point of the plate after impact is parallel to the z-axis and has the magnitude

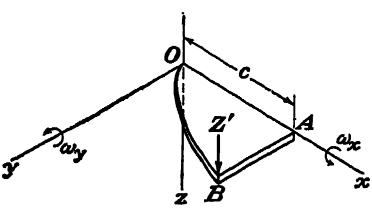


FIG. 97.

$$v' = \omega_x y - \omega_y x.$$

Thus, the last each of Eqs. (93) and (94) become

$$\left.\begin{aligned} \int_A \frac{w}{g}\,dA(\omega_x y - \omega_y x)y &= Z'\cdot\overline{AB}, \\ \int_A \frac{w}{g}\,dA(\omega_x y - \omega_y x)x &= -Z'\cdot\overline{OA}, \end{aligned}\right\} \tag{y}$$

where w is the weight per unit area of the plate. Taking $OA = c$, as

shown in the figure, and defining the parabola OB by the equation

$$y^2 = 4ax,$$

we find

$$\frac{w}{g}\int_A y^2 dA = \frac{16}{15}\frac{w}{g}a^{\frac{3}{2}}c^{\frac{5}{2}},$$

$$\frac{w}{g}\int_A x^2 dA = \frac{4}{7}\frac{w}{g}a\,c^{\frac{7}{2}},$$

$$\frac{w}{g}\int_A xy\,dA = \frac{2}{3}\frac{w}{g}ac^3,$$

and Eqs. (y) become

$$\left.\begin{aligned} \frac{w}{g}\left(\frac{16}{15}a^{\frac{3}{2}}c^{\frac{5}{2}}\omega_x - \frac{2}{3}ac^3\omega_y\right) &= Z'\sqrt{4ac}, \\ \frac{w}{g}\left(\frac{2}{3}ac^3\omega_x - \frac{4}{7}a\,c^{\frac{7}{2}}\omega_y\right) &= -Z'c. \end{aligned}\right\} \qquad (z)$$

Eliminating Z' from these equations, we can find the ratio ω_x/ω_y defining the direction of the axis of rotation of the plate immediately after the blow.

PROBLEMS

65. A homogeneous circular disk of radius a and mass M rotates with angular velocity ω about its geometric axis. If a point O in the circumference of the disk is suddenly fixed, find the new angular velocity ω' around this point and the tangential impulse Y' produced by fixing point O. *Ans.* $\omega' = \frac{1}{3}\omega$, $Y' = \frac{1}{3}Ma\omega$.

66. A homogeneous circular disk rotates in its own plane with angular velocity ω about a point A in its circumference. Suddenly, point A is released, and another point B in the circumference is fixed. Find the new angular velocity ω' if the arc AB subtends a central angle α.

Ans. $\omega' = \frac{1}{3}\omega(1 + 2\cos\alpha)$.

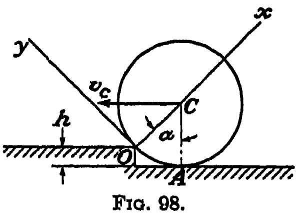


FIG. 98.

67. A homogeneous right circular cylinder of radius a and mass M rolls without slipping along a rough horizontal plane with velocity v_c. At a certain instant, it strikes an obstruction of height h, as shown in Fig. 98. Assuming that no slipping occurs at the point of impact O, find the components X' and Y' of the impulse. Assume inelastic impact.

$$Ans.\quad X' = Mv_c\sin\alpha,\ Y' = \frac{1}{3}Mv_c(1 - \cos\alpha),\ \alpha = \cos^{-1}\left(1 - \frac{h}{r}\right).$$

68. What is the condition governing the coefficient of friction μ in the preceding problem if there is to be no slipping at O? *Ans.* $\mu > (1 - \cos\alpha)/(3\sin\alpha)$.

69. A homogeneous right triangular plate ACB initially spins about its hypotenuse AC. Suddenly AC becomes free, and the edge BC is fixed. Prove that the angular velocity of the plate diminishes in the ratio $\overline{BC}/2\overline{AC}$.

70. A homogeneous rectangular plate $ABCD$ stands in a vertical plane with its horizontal lower edge AB, of length a, fixed in space. Owing to a slight disturbance,

the plate begins to fall under the influence of gravity, rotating about AB. When it reaches the horizontal, having acquired the angular velocity ω, AB becomes free and the edge AD, of length b, is suddenly fixed. Find the angular velocity ω' of the plate about AD just after impact. *Ans.* $\omega' = -(3a/4b)\sqrt{3g/a} = -(3a/4b)\omega$.

71. A solid homogeneous cube spins about a vertical diagonal with angular velocity ω. Find the new angular velocity ω' if one of its edges that does not meet the diagonal suddenly becomes fixed so that the cube starts to rotate with respect to this edge.

Ans. $\omega' = \omega/4\sqrt{3}$.

17. Balancing of a Single-cylinder Reciprocating Engine.—In discussing the balancing of reciprocating engines, we begin with a single-cylinder engine consisting of a crank OA, a connecting rod AB, and a piston B as shown in Fig. 99. We assume that this system is symmetrical

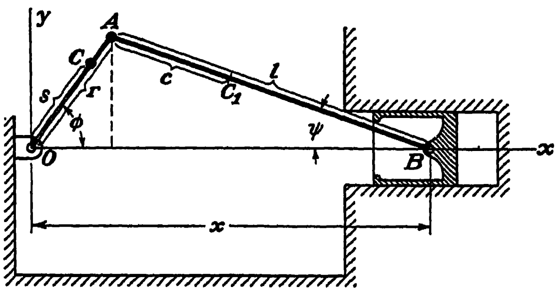


FIG. 99.

with respect to the plane of the figure; and through point O, we take the coordinate axes x and y as shown. In such a symmetrical case, the resultant of all external forces acting on the system will be completely defined by its components ΣX_e and ΣY_e and the moment $\Sigma(Y_e x - X_e y)$ with respect to the z-axis. We usually shall be more interested in the equal but opposite forces that the running engine exerts on its foundation; and for these quantities, we introduce the simplified notations

$$X = -\Sigma X_e, \qquad Y = -\Sigma Y_e, \qquad M_z = -\Sigma(Y_e x - X_e y). \qquad (a)$$

To find these forces when the motion of the system is known, the principles of linear and angular momentum will be useful. In our further discussion, we shall adopt the following notations (see Fig. 99):

r = length of crank,
l = length of connecting rod,
$s = OC$ = distance to center of gravity of crank,
$c = AC_1$ = distance to center of gravity of connecting rod,
m = mass of crank and crankpin,
m' = mass of connecting rod,
m'' = mass of piston,
ϕ = angle of rotation of crank, positive counterclockwise,

ψ = angle of rotation of connecting rod, positive clockwise,
x = distance OB.

Other notations will be introduced as the need for them arises.

As a preliminary step in this investigation, we shall find it convenient to replace the connecting rod by two equivalent masses m_1 and m_2' at its ends A and B, respectively. The sum of these two masses must equal the known mass m' of the rod; *i.e.*,

$$m_1 + m_2' = m',$$

and their mass-center must coincide with that of the rod; *i.e.*,

$$m_1 c = m_2'(l - c).$$

From these two conditions, we find

$$m_1 = m'\left(1 - \frac{c}{l}\right) \quad \text{and} \quad m_2' = m'\frac{c}{l}. \tag{b}$$

To the mass m_2' at B, we now add the mass m'' of the piston and denote the total joint mass by $m_2 = m_2' + m''$. Finally, we assume the mass m of the crank and crankpin to be concentrated at its center of gravity C. With these changes in the distribution of mass, we reduce the system of rigid bodies to a system of three mass particles m, m_1, and m_2, concentrated at C, A, and B, respectively. As far as the forces X and Y that the running engine transmits to its foundation are concerned, this fictitious system is dynamically equivalent to the actual system, since the mass-center is the same point in each case.

We proceed now to determine the forces X and Y. Noting that the velocities of points C, A, and B are $s\dot{\phi}$, $r\dot{\phi}$, and $\dot{x}$, respectively, and using the principle of linear momentum as represented by Eqs. (77) page 107, we may write, with due regard to signs,

$$\frac{d}{dt}(-ms\dot{\phi}\sin\phi - m_1 r\dot{\phi}\sin\phi + m_2\dot{x}) = \sum X_e = -X,$$

$$\frac{d}{dt}(ms\dot{\phi}\cos\phi + m_1 r\dot{\phi}\cos\phi) = \sum Y_e = -Y,$$

from which

$$\left.\begin{aligned} X &= (ms + m_1 r)(\dot{\phi}^2\cos\phi + \ddot{\phi}\sin\phi) - m_2\ddot{x}, \\ Y &= (ms + m_1 r)(\dot{\phi}^2\sin\phi - \ddot{\phi}\cos\phi). \end{aligned}\right\} \tag{95}$$

To represent Eqs. (95) in final form, we must express x in terms of the coordinate ϕ. Referring to Fig. 99, we see that

$$x = r\cos\phi + l\cos\psi \quad \text{and} \quad r\sin\phi = l\sin\psi.$$

Hence

$$x = r\cos\phi + l\sqrt{1 - \sin^2\psi} = r\cos\phi + l\sqrt{1 - \frac{r^2}{l^2}\sin^2\phi}.$$

Introducing the notation

$$\lambda = \frac{r}{l}, \tag{c}$$

the expression for x becomes

$$x = r\left(\cos\phi + \frac{1}{\lambda}\sqrt{1 - \lambda^2\sin^2\phi}\right). \tag{d}$$

In practical applications, λ^2 is usually a small quantity ($\frac{1}{6} < \lambda < 1/2.5$); and with the use of the binomial theorem, the second term in the parentheses of expression (d) can be expanded into a rapidly converging series and we obtain

$$x = r\left(\frac{1}{\lambda} + \cos\phi - \frac{\lambda}{2}\sin^2\phi - \frac{\lambda^3}{8}\sin^4\phi - \frac{\lambda^5}{16}\sin^6\phi - \cdots\right). \tag{e}$$

Observing that

$$\begin{aligned}\sin^2\phi &= \tfrac{1}{2}(1 - \cos 2\phi),\\ \sin^4\phi &= \tfrac{1}{8}(3 - 4\cos 2\phi + \cos 4\phi),\\ \sin^6\phi &= \tfrac{1}{32}(10 - 15\cos 2\phi + 6\cos 4\phi - \cos 6\phi),\end{aligned}$$

we finally obtain

$$x = r(A_0 + \cos\phi + \tfrac{1}{4}A_2\cos 2\phi - \tfrac{1}{16}A_4\cos 4\phi + \tfrac{1}{36}A_6\cos 6\phi - \cdots), \tag{96}$$

where

$$\left.\begin{aligned}A_0 &= 1/\lambda - \tfrac{1}{4}\lambda - \tfrac{3}{64}\lambda^3 - \tfrac{5}{256}\lambda^5 - \cdots.\\ A_2 &= \lambda + \tfrac{1}{4}\lambda^3 + \tfrac{15}{128}\lambda^5 + \cdots.\\ A_4 &= \tfrac{1}{4}\lambda^3 + \tfrac{3}{16}\lambda^5 + \cdots.\\ A_6 &= \tfrac{9}{128}\lambda^5 + \cdots.\end{aligned}\right\} \tag{f}$$

Substituting expression (96) for x into Eqs. (95), we obtain the following expressions for the forces that the running engine transmits to its foundation:

$$\left.\begin{aligned}X &= (ms + m_1r)(\dot\phi^2\cos\phi + \ddot\phi\sin\phi)\\ &\quad + m_2r\dot\phi^2(\cos\phi + A_2\cos 2\phi - A_4\cos 4\phi + A_6\cos 6\phi - \cdots)\\ &\quad + m_2r\ddot\phi(\sin\phi + \tfrac{1}{2}A_2\sin 2\phi - \tfrac{1}{4}A_4\sin 4\phi + \tfrac{1}{6}A_6\sin 6\phi - \cdots),\\ Y &= (ms + m_1r)(\dot\phi^2\sin\phi - \ddot\phi\cos\phi).\end{aligned}\right\} \tag{97}$$

TABLE XIII*

$\frac{1}{\lambda} = \frac{l}{r} =$	2.5	3	3.5	4	4.5	5	5.5	6
A_2	0.4173	0.3431	0.2918	0.2540	0.2250	0.2020	0.1833	0.1678
A_4	0.0182	0.0101	0.0062	0.0041	0.0028	0.0021	0.0015	0.0012
A_6	0.0009	0.0003	0.0001	0.0001				

* This table has been taken from Biezeno and Grammel, "Technische Dynamik," Springer, 1939. This book contains a very complete discussion of engine balancing.

For various values of the ratio λ, the values of the coefficients A_i as given by expressions (*f*) are shown in Table XIII. We see that the terms in the series of Eqs. (97) are decreasing rapidly, especially for small values of λ. Thus a satisfactory accuracy can be obtained by keeping only a few terms of these series.

In discussing engine balance, we usually are most interested in the case where the engine runs with uniform angular velocity ω. For this condition, the terms in Eqs. (97) that contain $\ddot{\phi}$ vanish; and putting $\dot{\phi} = \omega$, we obtain

$$\left.\begin{aligned} X &= \omega^2[(ms + m_1r + m_2r)\cos\phi + m_2r(A_2\cos 2\phi - A_4\cos 4\phi \\ &\qquad\qquad\qquad\qquad\qquad + A_6\cos 6\phi - \cdots)]. \\ Y &= \omega^2(ms + m_1r)\sin\phi. \end{aligned}\right\} \quad (98)$$

We observe now that an engine which does not transmit forces X and Y to its foundation must satisfy two conditions, namely:

$$ms + m_1r = 0 \qquad \text{and} \qquad m_2 = 0. \qquad (g)$$

The first of these conditions is readily satisfied by introducing counterweights on the crankshaft as shown in Fig. 100. The size of the counterweight must be such that the center of gravity of the total mass m of crank and counterweight together will be opposite the crankpin and at such a distance s from the axis of rotation that

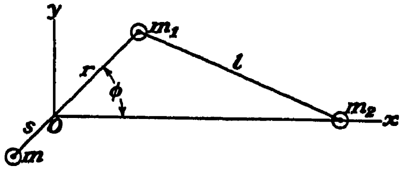


FIG. 100.

$$s = \frac{m_1}{m} r. \qquad (h)$$

When this condition is satisfied, the mass-center of m and m_1 remains stationary at O; and from the second of Eqs. (98), we see that the transverse component Y of the disturbing force vanishes.

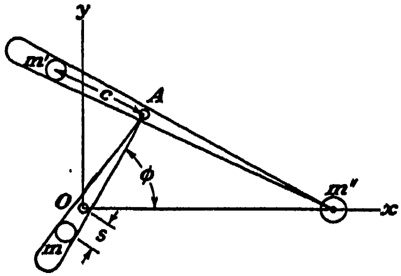


FIG. 101.

To satisfy the second of conditions (*g*), the mass m_2 must vanish. This will be possible only if the portion m_2' contributed by the connecting rod is negative and equal to the mass m'' of the piston. This can be accomplished by extending the connecting rod beyond the crankpin A so that the distance c will be negative as shown in Fig. 101. Using the

second of expressions (b) above, we see that the required distance c will be

$$c = \frac{m''}{m'} l. \tag{i}$$

When conditions (h) and (i) are satisfied simultaneously, the mass-center of m, m', and m'' remains stationary at O and the forces X and Y both vanish. Needless to say, the method of balancing indicated in Fig. 101 is entirely impracticable, since it requires too great an extension of the connecting rod beyond the crankpin. More practicable methods of balancing will be considered at the end of this article.

To complete our investigation of the unbalance in a one cylinder engine, we have to consider now the moment of the forces X and Y with respect to the z-axis. For this purpose, we need the general expression for the corresponding angular momentum of the system in Fig. 99. The piston mass m'' moves along the x-axis; the moment of its momentum with respect to point O vanishes; and we do not need to consider it further. The angular momentum of the crank with respect to its axis of rotation is

$$mi_0{}^2\dot{\phi}, \tag{j}$$

where i_0 is the radius of gyration with respect to the z-axis. In considering the connecting rod, we recall that it was replaced by two mass particles at its ends A and B [see Eqs. (b)]. As far as translation of the connecting rod was concerned, these two mass particles were dynamically equivalent to the actual rod. To be dynamically equivalent in rotation, the two particles at the ends of the line AB must have the same moment of inertia as the actual rod about any axis normal to the plane of the figure. In general, the masses m_1 and m_2' as defined by Eqs. (b) will not satisfy this requirement, and we have to assign to one of them, say the mass m_1 at A, a fictitious moment of inertia I such that, for total moment of inertia with respect to point B,

$$m_1 l^2 + I = I' + m'(l - c)^2,$$

where I' is the moment of inertia of the actual connecting rod about its centroidal axis normal to the xy-plane. From this condition and the first of Eqs. (b), defining the magnitude of m_1, we find, for the required fictitious moment of inertia of the particle at A,*

$$I = I' - m'c(l - c). \tag{k}$$

Finally, since positive ψ in Fig. 99 is measured clockwise while positive ϕ is measured counterclockwise, the angular velocity of the connecting rod and, hence, also of the particle m_1 at A is $-\dot{\psi}$. Thus the angular

* This fictitious moment of inertia may be either positive or negative.

momentum of this particle with respect to the z-axis is

$$m_1 r^2 \dot{\phi} - I\dot{\psi}. \qquad (l)$$

The particle m_2' at B has no moment of inertia and moves along the x-axis; thus its moment of momentum with respect to the z-axis vanishes along with that of the piston m''. We conclude then that the total angular momentum of the system with respect to the z-axis will be obtained as the sum of expressions (j) and (l). Equating the rate of change of this angular momentum to the corresponding moment of all external forces acting on the system, we obtain

$$\frac{d}{dt}(mi_0^2\dot{\phi} + m_1 r^2\dot{\phi} - I\dot{\psi}) = \sum (Y_e x - X_e y) = -M_z,$$

from which

$$M_z = -(mi_0^2 + m_1 r^2)\ddot{\phi} + I\ddot{\psi}. \qquad (99)$$

To represent Eq. (99) in final form, we must express $\ddot{\psi}$ in terms of ϕ and its derivatives. For this purpose, we use the relationship

$$r \sin \phi = l \sin \psi,$$

from which

$$\cos \psi = \sqrt{1 - \frac{r^2}{l^2}\sin^2\phi}.$$

Expanding this by the binomial theorem and using notation (c), we get

$$\cos \psi = 1 - \frac{\lambda^2}{2}\sin^2\phi - \frac{\lambda^4}{8}\sin^4\phi - \frac{\lambda^6}{16}\sin^6\phi - \cdots.$$

Differentiating with respect to time and dividing by

$$\sin \psi = \lambda \sin \phi,$$

we find

$$\dot{\psi} = \dot{\phi}(\lambda \cos \phi + \tfrac{1}{2}\lambda^3 \sin^2\phi \cos \phi + \tfrac{3}{8}\lambda^5 \sin^4\phi \cos \phi + \cdots).$$

Substituting for $\sin^2\phi$, $\sin^4\phi$, . . . their values from page 138 and using the formula

$$\cos 2k\phi \cos \phi = \tfrac{1}{2}\cos(2k+1)\phi + \tfrac{1}{2}\cos(2k-1)\phi,$$

we obtain

$$\dot{\psi} = \lambda\dot{\phi}(C_1 \cos \phi - \tfrac{1}{3}C_3 \cos 3\phi + \tfrac{1}{5}C_5 \cos 5\phi - + \cdots), \qquad (m)$$

where

$$\left.\begin{aligned} C_1 &= 1 + \tfrac{1}{8}\lambda^2 + \tfrac{3}{64}\lambda^4 + \cdots, \\ C_3 &= \tfrac{3}{8}\lambda^2 + \tfrac{27}{128}\lambda^4 + \cdots, \\ C_5 &= \tfrac{15}{128}\lambda^4 + \cdots. \end{aligned}\right\} \qquad (n)$$

The numerical values of these coefficients for various values of λ are given in Table XIV below.

TABLE XIV*

$\frac{1}{\lambda} = \frac{l}{r} =$	2.5	3	3.5	4	4.5	5	5.5	6
C_1	1.021	1.014	1.010	1.008	1.006	1.005	1.004	1.003
C_3	0.066	0.044	0.032	0.024	0.019	0.015	0.013	0.011
C_5	0.004	0.002	0.001	0.001				

* Taken from Biezeno and Grammel "Technische Dynamik," Springer, 1939.

Differentiating expression (m) with respect to time, we obtain

$$\ddot{\psi} = -\lambda\dot{\phi}^2(C_1\sin\phi - C_3\sin 3\phi + C_5\sin 5\phi - + \cdots) + \lambda\ddot{\phi}(C_1\cos\phi - \tfrac{1}{3}C_3\cos 3\phi + \tfrac{1}{5}C_5\cos 5\phi - + \cdots).$$

Substituting this expression for $\ddot{\psi}$ in Eq. (99), we get finally

$$M_z = -I\lambda\dot{\phi}^2(C_1\sin\phi - C_3\sin 3\phi + C_5\sin 5\phi - + \cdots) + I\lambda\ddot{\phi}(C_1\cos\phi - \tfrac{1}{3}C_3\cos 3\phi + \tfrac{1}{5}C_5\cos 5\phi - + \cdots) - (mi_0^2 + m_1r^2)\ddot{\phi}. \quad (100)$$

We see that this moment vanishes for any configuration of the system in Fig. 99 only if

$$I = 0 \quad \text{and} \quad mi_0^2 + m_1r^2 = 0. \qquad (o)$$

These two conditions cannot be satisfied simultaneously. As may be seen from expression (k), the first requires that $c < l$, while the second can be satisfied only if $m_1 < 0$ which, as can be seen from the first of expressions (b), requires that $c > l$. Thus we can never entirely eliminate M_z by any kind of balancing. Practically, however, fluctuations in the angular velocity $\dot{\phi}$ of the engine are usually very small, so that the terms multiplied by $\ddot{\phi}$ in Eq. (100) are only of secondary importance as compared with the terms multiplied by $\dot{\phi}^2$. This is especially true for modern high-speed engines in which $\dot{\phi}^2$ is always a large number, and we obtain a quiet-running engine by satisfying only the first of conditions (o). This, as we see from expression (k), requires

$$I' = m'c(l - c) = 0.$$

Denoting by i_1 the radius of gyration of the connecting rod corresponding to the moment of inertia I', this reduces to

$$i_1^2 = c(l - c),$$

which can be expressed in either of the following two forms:

$$\frac{i_1^2 + c^2}{c} = l \quad \text{or} \quad \frac{i_1^2 + (l - c)^2}{l - c} = l. \qquad (p)$$

These equations indicate that the connecting rod must be so proportioned that when suspended either from A or B, we obtain a physical pendulum of effective length l.* This requirement can easily be satisfied by slight extensions beyond the crank and crosshead bearings as shown in Fig. 102. With this form of connecting rod, the principal part of the disturbing moment M_z will be eliminated.

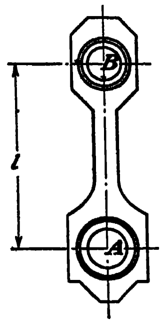


FIG. 102.

From all of the foregoing discussion, we conclude that a single-cylinder reciprocating engine cannot be perfectly balanced and that it always produces some action on its foundation. Assuming a uniform angular velocity and giving to the connecting rod a form satisfying Eqs. (p), we eliminate the worst part of the unbalanced moment M_z. By proper counterweights on the crankshaft, we can also eliminate completely the transverse force Y [see last of Eqs. (98)]. The elimination of the longitudinal force X by extension of the connecting rod beyond the crankpin as discussed on page 139 is impracticable, but this force can be eliminated in other ways. Assume that the first of conditions (g) is satisfied so that the transverse force Y vanishes. Then from the first of Eqs. (98), we obtain

$$X = m_2\omega^2 r[\cos\phi + (A_2\cos 2\phi - A_4\cos 4\phi + A_6\cos 6\phi - + \cdots)]. \quad (q)$$

From Table XIII, we see that the coefficients A_i in this expression are rapidly decreasing with increasing subscripts and only the first few terms need be considered. If, by some arrangement, we eliminate the term $\cos\phi$, we obtain balancing of the first order; if we eliminate also the term $\cos 2\phi$, we obtain balancing of the second order, etc.

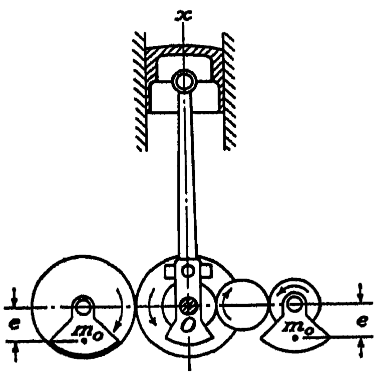


FIG. 103.

Balancing of the first order can be accomplished by introducing additional rotating masses m_0 as shown in Fig. 103. By a proper system of gears, these masses are connected with the crankshaft O and have the same angular velocity ω that it has. The eccentricities of these masses make the angle π with the crank so that when the crank makes any angle ϕ with the x-axis, they make therewith the angles $\pi - \phi$ and $\pi + \phi$, respectively. Thus the centrifugal forces $m_0\omega^2 e$ of the masses m_0 have, in the direction of the x-axis, the resultant

* See the authors' "Engineering Mechanics," 2d ed., p. 401.

$$m_0\omega^2 e[\cos(\pi - \phi) + \cos(\pi + \phi)] = -2m_0\omega^2 e \cos \phi. \tag{r}$$

From Eq. (q), it can now be seen that balancing of the first order will be attained if

$$m_2\omega^2 r \cos \phi - 2m_0\omega^2 e \cos \phi = 0,$$

where $m_2 = m_2' + m''$. The value of m_2' is given by the second of Eqs. (b), and we obtain

$$\frac{2m_0 e}{r} = m' \frac{c}{l} + m'', \tag{s}$$

which defines the proper countermasses for first-order balancing. We note that these countermasses are so arranged that their transverse inertia forces are self-balancing.

An arrangement similar to that in Fig. 103 can be used for balancing the forces of second order. In this case, we introduce such a system of gears that the angular velocities of the countermasses m_0 are equal to 2ω. Then the resultant centrifugal force of these masses in the x-direction is

$$4m_0\omega^2 e[\cos(\pi - 2\phi) + \cos(\pi + 2\phi)] = -8m_0\omega^2 e \cos 2\phi.$$

Thus, from Eq. (q), we conclude that second-order balancing will be realized if

$$m_2\omega^2 r A_2 \cos 2\phi - 8m_0\omega^2 e \cos 2\phi = 0,$$

and the required countermasses are defined by the relation

$$\frac{8m_0 e}{r} = A_2\left(m' \frac{c}{l} + m''\right). \tag{t}$$

Since, from Eqs. (f), $A_2 \approx \lambda = r/l$, second-order balancing will require comparatively small countermasses.

Such arrangements as that illustrated in Fig. 103 are complicated, especially if we try to attain balancing of higher order than the first. A more satisfactory and efficient method of balancing is shown in Fig. 104. Briefly, it consists of introducing, in the crankcase of the engine, two identical dummy cranks, rods, and pistons so arranged that they always set up disturbing forces equal and opposite to those set up by the master crank, rod, and piston. When the dummy cranks make the angle π with the master crank, it is evident that the forces set up by the two dummy systems are always opposite to those set up by the master system. Regarding their magnitudes, we refer to Eqs. (98) together with expressions (b) and (f), from which we conclude that if the linear dimensions of the reciprocating system in Fig. 99 are all decreased in a constant ratio α while the masses m, m', and m'' are decreased in a constant ratio β, then the forces X and Y as defined by Eqs. (98) are decreased in the

ratio $\alpha\beta$. Returning now to Fig. 104, let quantities with the zero subscript refer to the dummy systems and those without subscript to the master system. Then if $r_0 = \alpha r$, $s_0 = \alpha s$, $l_0 = \alpha l$, and $c_0 = \alpha c$ while $m_0 = \beta m$, $m_0' = \beta m'$, and $m_0'' = \beta m''$, we conclude that the forces set up by the two dummy systems together will be

$$X_0 = -2\alpha\beta X \quad \text{and} \quad Y_0 = -2\alpha\beta Y,$$

where X and Y are the forces set up by the master system. Making $\alpha\beta = \frac{1}{2}$, we obtain perfect balance with regard to forces. Thus, for example, if each dummy system has masses equal to the corresponding masses of the master system but lengths only half as great, we have the desired result.

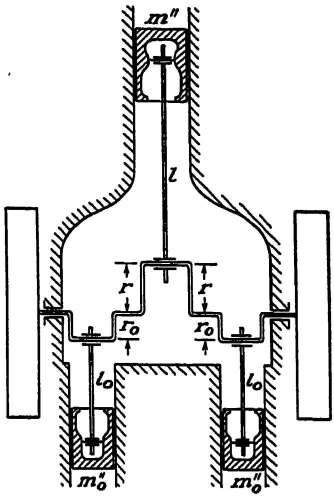


FIG. 104.

It is left as an exercise for the student to show that in the case of the engine in Fig. 104, the above conditions do not result in a vanishing moment M_s. However, by using connecting rods like the one shown in Fig. 102 and heavy flywheels to ensure nearly constant angular velocity ω, we obtain an almost perfectly balanced engine.

PROBLEMS

72. For the single-cylinder engine shown in Fig. 99, the following numerical data are given: $m' = 3.60$ lb., $m'' = 3.84$ lb., $r = 2.50$ in., $l = 11.00$ in., $c = 3.25$ in. The crankshaft alone has already been balanced; *i.e.*, $s = 0$. Calculate the maximum values of the periodic forces X and Y that the engine transmits to its foundation when running uniformly at 500 r.p.m.

Ans. $X_{\max} = 153.8$ lb.; $Y_{\max} = 46.3$ lb.

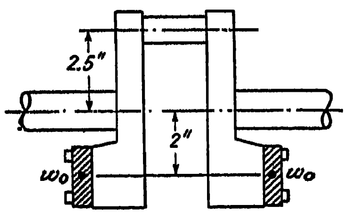


FIG. 105.

73. Referring to Fig. 105, what additional counterweights w_0 with eccentricity $e = 2$ in. must be added to the crankshaft of the engine described in the preceding problem in order to eliminate the transverse force Y?

Ans. $w_0 = 1.625$ lb.

74. Assuming that we also desire to attain first-order balancing with regard to X of the engine described in Prob. 72 by using the scheme shown in Fig. 103, what additional rotating countermasses m_0 with eccentricity $e = 2$ in. will be required?

Ans. $gm_0 = 9.20$ lb.

75. When suspended as a physical pendulum from the circumference of the wrist-pin bearing of diameter $d = 0.875$ in., the connecting rod of the engine described in Prob. 72 is observed to have a period $\tau = 1.023$ sec. Calculate the maximum value of the unbalanced moment M_s when the engine runs at 500 r.p.m.

Ans. $(M_s)_{\max} = 54.25$ in.-lb.

18. Balancing of Multicylinder Reciprocating Engines.—The equations developed in the preceding article for a single cylinder can be used in discussing the balance of various multicylinder reciprocating engines. Very often, such engines consist of several identical parallel cylinders arranged in series in one plane and working on a common crankshaft. By a proper arrangement of these cylinders as to spacing and crank angles, it is often possible to attain a partial or even complete balancing of the engine at constant speed.

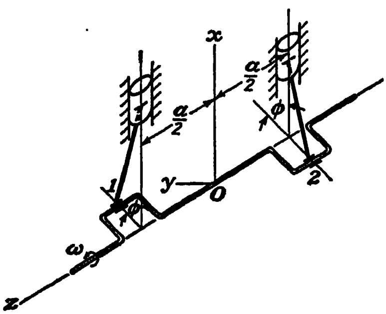


FIG. 106.

Consider, first, the case of a two-cylinder engine with cranks 180 deg. apart as shown in Fig. 106. Assuming uniform angular velocity ω and taking coordinate axes as shown, the forces X and Y and the moment M_z for cylinder 1 are obtained by using Eqs. (98) and (100). The corresponding forces for cylinder 2 are obtained by substituting in the same equations $\phi + \pi$ for ϕ. Thus, for both cylinders together, we obtain

$$X = 2\omega^2 m_2 r(A_2\cos 2\phi - A_4\cos 4\phi + A_6\cos 6\phi - + \cdots),$$
$$Y = 0, \qquad M_z = 0. \qquad (101)$$

We see that with such an arrangement, the transverse force Y and the moment M_z are automatically eliminated. Also the longitudinal force X of the first order is balanced and only those of higher order remain.

As long as we had a single-cylinder engine, all forces were in one plane and were completely defined by the components X, Y, and M_z. Now, with two cylinders, the forces are no longer in one plane, and we must consider also their moments M_x and M_y with respect to the x- and y-axes. Denoting the cylinder spacing by a and using Eqs. (98), we find

$$\left.\begin{aligned}
M_x &= \tfrac{1}{2}\omega^2 a[-(ms + m_1 r)\sin\phi + (ms + m_1 r)\sin(\phi + \pi)] \\
&= \underline{-\omega^2 a(ms + m_1 r)\sin\phi,} \\
M_y &= \tfrac{1}{2}\omega^2 a\{(ms + m_1 r + m_2 r)\cos\phi + m_2 r(A_2\cos 2\phi - A_4\cos 4\phi \\
&\quad + A_6\cos 6\phi - + \cdots) - (ms + m_1 r + m_2 r)\cos(\phi + \pi) \\
&\quad - m_2 r[A_2\cos 2(\phi + \pi) - A_4\cos 4(\phi + \pi) \\
&\quad + A_6\cos 6(\phi + \pi) - + \cdots]\} \\
&= \underline{+\omega^2 a(ms + m_1 r + m_2 r)\cos\phi.}
\end{aligned}\right\} \qquad (102)$$

We note that all terms of higher order in these expressions vanish, leaving only unbalanced moments of the first order. These moments disappear completely only if conditions (g), page 139, are fulfilled. The first of

these conditions is usually satisfied without much difficulty by overbalancing the crankshaft as already discussed, but the second condition is hard to fulfill, and the moments M_x and M_y usually will remain incompletely balanced.

A complete balancing of the moments M_x and M_y can be accomplished by introducing additional rotating masses m_0 arranged as shown in Fig. 107. Two pairs of such masses located symmetrically with respect to the xz-plane and rotating with the angular velocity ω of the engine produce a couple the moment of which with respect to the y-axis is

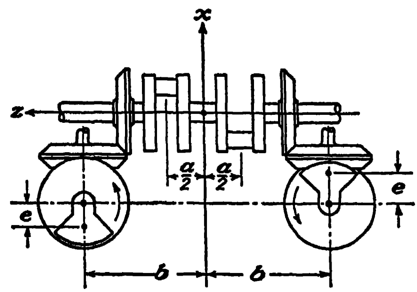


FIG. 107.

$$-2m_0e2b\omega^2\cos\phi.$$

Assuming that the moment M_x has already been balanced by satisfying the first of conditions (g), page 139, we see now that the remainder of M_y will be balanced by selecting the masses m_0 so that

$$-2m_0e2b\omega^2\cos\phi + \omega^2am_2r\cos\phi = 0,$$

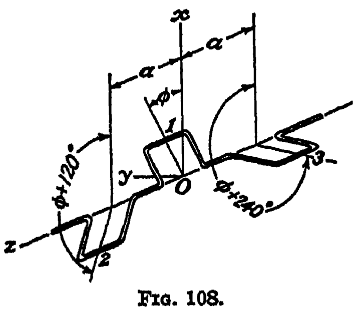


FIG. 108.

from which

$$m_0 = \frac{ar}{4be}m_2, \tag{a}$$

where $m_2 = m_2' + m''$ (see page 137).

We consider next the case of a three-cylinder engine with the crank arrangement shown in Fig. 108. Selecting coordinate axes as shown, we use, for the middle crank 1, Eqs. (98) and (100) as before. For the cranks 2 and 3, we substitute for ϕ in these same equations the values $\phi + 120$ deg. and $\phi + 240$ deg., respectively. Then superimposing the forces for all three cylinders and confining our attention to a uniform engine speed, we obtain

$$\left.\begin{aligned}
X &= 3m_2\omega^2r(A_6\cos 6\phi - A_{12}\cos 12\phi + \cdots),\\
Y &= 0,\\
M_x &= -\sqrt{3}\,\omega^2a(ms + m_1r)\cos\phi,\\
M_y &= -\sqrt{3}\,\omega^2a[(ms + m_1r + m_2r)\sin\phi - m_2r(A_2\sin 2\phi\\
&\qquad + A_4\sin 4\phi - A_8\sin 8\phi - A_{10}\sin 10\phi + \cdots)],\\
M_z &= 3\omega^2I\lambda(C_3\sin 3\phi - C_9\sin 9\phi + C_{15}\sin 15\phi - + \cdots).
\end{aligned}\right\} \tag{103}$$

It will be seen that for this arrangement, the transverse force Y vanishes completely while the longitudinal force up to the sixth order is also balanced. The moment M_z can always be balanced by making $I = 0$ (see page 142). Thus only the moments M_x and M_y will give any trouble, and a partial balancing of these can always be arranged as in the case of the two-cylinder engine discussed above (see Fig. 107).

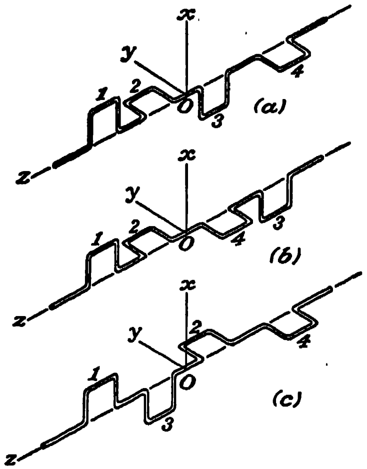


FIG. 109.

In discussing the balance of a four-cylinder engine with uniform cylinder spacing a and constant angular velocity ω, we consider three possible crank arrangements as shown in Fig. 109. In each case, the cranks are numbered in the order in which they follow each other 90 deg. apart. Using Eqs. (98) and (100) and substituting for ϕ: ϕ, $\phi + 90$ deg., $\phi + 180$ deg., and $\phi + 270$ deg., for cranks 1, 2, 3, and 4, respectively, we obtain, after summation,

$$\left.\begin{aligned} X &= -4m_2\omega^2 r(A_4\cos 4\phi + A_8\cos 8\phi + \cdots), \\ Y &= 0, \\ M_z &= 0. \end{aligned}\right\} \tag{104}$$

We see that in each case, the transverse force Y and the moment M_z are completely balanced while the longitudinal force X is balanced up to the fourth order.

The expressions for the moments M_x and M_y will be different for each of the arrangements in Fig. 109 and can be presented as follows: For case (a),

$$\left.\begin{aligned} M_x &= -\sqrt{8}\,\omega^2 a(ms + m_1 r)\sin(\phi + 45°), \\ M_y &= +\sqrt{8}\,\omega^2 a(ms + m_1 r + m_2 r)\cos(\phi + 45°) \\ &\quad + 2m_2\omega^2 ra(A_2\cos 2\phi + A_6\cos 6\phi + \cdots). \end{aligned}\right\} \tag{104a}$$

For case (b), with $\beta = \tan^{-1}(\frac{1}{3}) = 18.4$ deg.

$$\left.\begin{aligned} M_x &= -\sqrt{10}\,\omega^2 a(ms + m_1 r)\sin(\phi + \beta), \\ M_y &= +\sqrt{10}\,\omega^2 a(ms + m_1 r + m_2 r)\cos(\phi + \beta). \end{aligned}\right\} \tag{104b}$$

For case (c),

$$\left.\begin{aligned} M_x &= -\sqrt{2}\,\omega^2 a(ms + m_1 r)\sin(\phi + 45°), \\ M_y &= +\sqrt{2}\,\omega^2 a(ms + m_1 r + m_2 r)\cos(\phi + 45°) \\ &\quad + 4m_2\omega^2 ra(A_2\cos 2\phi + A_6\cos 6\phi + \cdots). \end{aligned}\right\} \tag{104c}$$

We see that regarding moments M_x and M_y of the first order, the arrangement in Fig. 109*c* is the most favorable. Sometimes, however, the arrangement in Fig. 109*b* is preferred, since it contains only first-order moments which can be completely balanced by making $ms + m_1r = 0$ and then introducing rotating masses m_0 as shown in Fig. 107.

In dealing with multicylinder engines, a graphical method of representation of the forces and moments for individual cylinders is sometimes

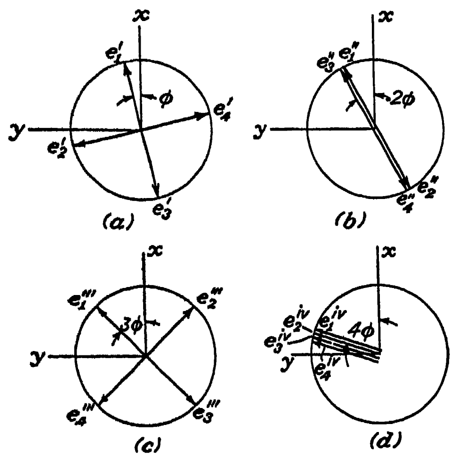


FIG. 110.

used. For this purpose we introduce unit vectors e_1', e_2', . . . with angles of rotation ϕ_1, ϕ_2, . . . to represent forces of the first order; likewise, unit vectors e_1'', e_2'', . . . with angles of rotation $2\phi_1$, $2\phi_2$, . . . to represent forces of the second order; etc. Then the geometric sums of these vectors, multiplied by proper factors, give the combined effect of all cylinders. Consider, for example, the longitudinal force X of a four-cylinder engine (Fig. 109*a*). The force produced by the first cylinder is given by the first of Eqs. (98). Using the above vector notations, the result of summation of the actions of all four cylinders will be

$$X = \omega^2[(ms + m_1r + m_2r)\Sigma_x e_i' + m_2r(A_2\Sigma_x e_i'' - A_4\Sigma_x e_i^{\mathrm{IV}} + A_6\Sigma_x e_i^{\mathrm{VI}} - + \cdots)],$$

where the summation signs are understood to indicate the x-projections of the geometric sums of the unit vectors. The vectors e_i' of the first order are shown in Fig. 110*a*, and we see, as already noted in Eqs. (104), that the forces of the first order are balanced. The same conclusion can be made regarding forces of the second and third orders so shown in Fig. 110*b* and 110*c*. But the forces of the fourth order are unbalanced as can be seen from Fig. 110*d*. The sum of their projections on the

x-axis is given by the first term of the series (104). A similar method of construction can be developed for the graphical summation of moments M_x, and M_y, it being necessary only to multiply each unit vector by its distance from the origin of coordinates.

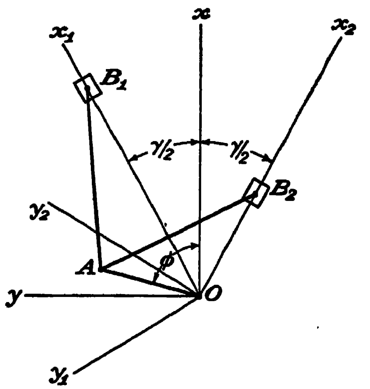


FIG. 111.

We shall now discuss briefly the case of a V-type engine in which the axes of the cylinders are not parallel. We begin with two identical coplaner cylinders having between their axes the angle γ and operating from a common crank OA as shown in Fig. 111. Confining our attention to uniform angular velocity ω of the crank, we can find the longitudinal and transverse forces for each cylinder by using Eqs. (98). It is only necessary to use in these expressions $m/2$ instead of m, since the crank mass must be equally divided between the two cylinders. To represent the combined action of the two cylinders, we choose, in their common plane, the coordinate axes x and y as shown, the x-axis bisecting the angle γ. Then the components of the resultant force are

$$X = (X_1 + X_2)\cos\frac{\gamma}{2} - (Y_1 - Y_2)\sin\frac{\gamma}{2},$$

$$Y = (X_1 - X_2)\sin\frac{\gamma}{2} + (Y_1 + Y_2)\cos\frac{\gamma}{2}.$$

The values of X_1, Y_1 and X_2, Y_2 are found from Eqs. (98) by substituting $\phi_1 = \phi - (\gamma/2)$ and $\phi_2 = \phi + (\gamma/2)$, and we obtain

$$\left.\begin{aligned} X &= \omega^2 \Big\{[ms + 2m_1r + m_2r(1 + \cos\gamma)]\cos\phi \\ &\quad + 2m_2r\cos\frac{\gamma}{2}(A_2\cos\gamma\cos 2\phi - A_4\cos 2\gamma\cos 4\phi + \cdots)\Big\}, \\ Y &= \omega^2 \Big\{[ms + 2m_1r + m_2r(1 - \cos\gamma)]\sin\phi \\ &\quad + 2m_2r\sin\frac{\gamma}{2}(A_2\sin\gamma\sin 2\phi - A_4\sin 2\gamma\sin 4\phi + \cdots)\Big\}. \end{aligned}\right\} \quad (105)$$

Using Eq. (100) in the same way and neglecting terms containing $\dot{\phi}$, we have for the combined moment about the z-axis,

$$M_z = -2\omega^2 I\lambda\left(C_1\cos\frac{\gamma}{2}\sin\phi - C_3\cos\frac{3\gamma}{2}\sin 3\phi + \cdots\right) \quad (106)$$

As would be expected, the moment M_z can be made to vanish if the connecting rods are of the form shown in Fig. 102 so that $I = 0$.

Regarding the longitudinal and transverse forces X and Y, as defined by expressions (105), we see that in general, they are unbalanced. The terms representing the most important forces of the first order can be made to vanish if we have

$$ms + 2m_1r + m_2r(1 + \cos\gamma) = 0,$$
$$ms + 2m_1r + m_2r(1 - \cos\gamma) = 0,$$

which requires that

$$\left.\begin{aligned} ms + 2m_1r + m_2r &= 0, \\ \cos\gamma &= 0, \end{aligned}\right\} \tag{b}$$

The first of these requirements can be satisfied by counterbalancing the crankshaft, and the second by making $\gamma = 90$ deg. It will be seen that

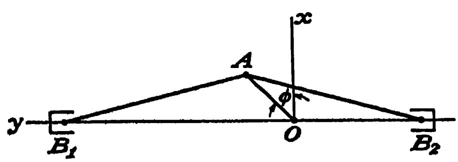


FIG. 112.

when these conditions are satisfied, some of the forces of higher order in expressions (105) will also vanish.

For the two-cylinder engine in Fig. 112 where $\gamma = 180$ deg., Eqs. (105) and (106) above reduce to

$$\left.\begin{aligned} X &= \omega^2(ms + 2m_1r)\cos\phi, \\ Y &= \omega^2(ms + 2m_1r + 2m_2r)\sin\phi, \\ M_z &= 0. \end{aligned}\right\} \tag{107}$$

The moments M_z and all forces of higher order than the first are balanced, only forces of the first order remaining. By counterbalancing the crankshaft, either the force X or the force Y can be balanced, but not both. If X is eliminated by making $ms + 2m_1r = 0$, the remaining part of Y can be balanced by using additional rotating masses m_0 as shown in Fig. 103, and the system is completely balanced.

If several identical cylinders with equal angles γ between their axes are arranged radially around a single crankshaft like the two in Fig. 111, it can be shown that to balance the forces of the first order in such case, we have, instead of Eqs. (b), the following conditions to fulfill.

$$\left.\begin{aligned} ms + \frac{nr}{2}(2m_1 + m_2) &= 0, \\ \gamma &= \frac{360°}{n}, \end{aligned}\right\} \tag{c}$$

where n is the number of cylinders. The first of these conditions can always be satisfied by a proper counterbalancing of the crank so that s is negative, which may, of course, require rather heavy counterweights. The second condition is always easy to fulfill. It may be noted, for example, that the system in Fig. 112 satisfies this condition for the particular case where $n = 2$.

If conditions (*c*) are satisfied and the number of cylinders n is even, the forces of higher order will also be balanced. If n is an odd number, the forces are balanced up to the order $n - 1$. The moments M_z can always be balanced by making $I = 0$ (see page 142). Assuming that forces of the sixth and higher order are negligible, it can be stated that for such radial engines with six or more cylinders, we can always attain good balance.

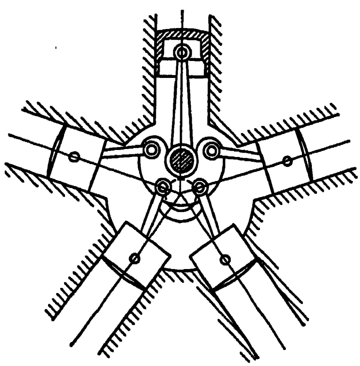


FIG. 113.

In practical applications, it is usually impossible to attach all the connecting rods to the same crankpin, and the arrangement shown in Fig. 113 is used. One connecting rod, called the master rod, is attached to the crankpin, and the others are attached to it. Our preceding discussion applies to this case as a good first approximation. An accurate calculation of the unbalanced forces requires a more complicated analysis.[1]

PROBLEMS

76. Show that at constant speed, a four-cylinder engine with the crankshaft arrangement shown in Fig. 114 will be balanced as regards both first-order forces X and Y and first-order moments M_x and M_y.

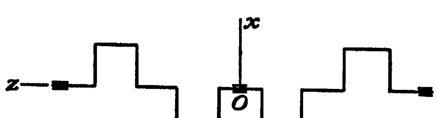


FIG. 114.

77. Using such vector diagrams as those shown in Fig. 110, show that a six-cylinder engine with cranks arranged in the successive angular positions 0, 120, 240, 240, 120, 0 deg. is completely balanced at constant speed up to the sixth order as regards both the forces X and Y and their moments M_x and M_y. See Fig. 108 for one-half of the crankshaft.

78. Investigate the state of balance, at constant speed, of a straight-eight engine with cranks arranged in the successive angular positions 0, 180, 90, 270, 270, 90, 180, 0 deg. See Fig. 109*c* for one-half of the crankshaft.

[1] See paper by P. Riekert, *Ing. Arch.*, vol. I, pp. 16, 245, 1930.

79. Show that at uniform speed, any straight engine with n identical parallel cylinders in the xz-plane and uniform angular spacing $360/n$ deg of the cranks is balanced at least up to the nth order as regards forces X and Y and their moments M_z.

80. Assuming that conditions (b), p. 151, are satisfied for each pair of cylinders of a V-8 engine having the crankshaft shown in Fig. 109b, investigate its state of balance, at constant speed, as regards moments M_x and M_y.

19. Kinetic Energy and Work.—The notions of momentum and impulse used throughout the preceding articles have been found very helpful in discussing the dynamics of various systems of particles. We turn our attention now to the equally important notions of kinetic energy and work. In discussing these concepts, we begin with a single particle of mass m moving with velocity v under the action of a resultant force F (Fig. 115). The scalar quantity

$$\tfrac{1}{2}mv^2 = \tfrac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2), \qquad (a)$$

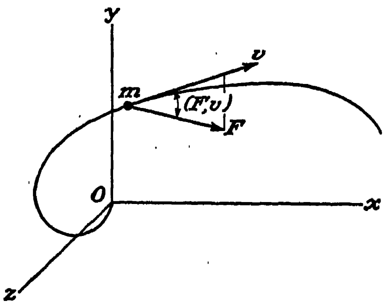


FIG. 115.

where $\dot{x}$, $\dot{y}$, $\dot{z}$ are the projections of the velocity vector v, is called the *kinetic energy* of the particle.[1] Now let the particle move a small distance ds along its path. Then the *work* done by the resultant force F is

$$F \cos (F,v)ds = X\,dx + Y\,dy + Z\,dz, \qquad (b)$$

where dx, dy, dz are the projections of the displacement ds and (F, v) is the angle that F makes with the direction of motion. Using the equations of motion

$$m\ddot{x} = X, \qquad m\ddot{y} = Y, \qquad m\ddot{z} = Z,$$

we can readily establish the relationship[2]

$$d(\tfrac{1}{2}mv^2) = F \cos(F,v)ds. \qquad (c)$$

This equation states that during motion of the particle, the change in kinetic energy from one position to an adjacent position is equal to the work of the resultant force on that displacement.

In the case of a system of particles, we distinguish between external forces F_e and internal forces F_i acting on any particle. Thus for one particle of a system, we write

$$d(\tfrac{1}{2}mv^2) = F_e\cos (F_e,v)ds + F_i\cos(F_i,v)ds. \qquad (d)$$

[1] Prior to 1829, the quantity mv^2, called *vis viva*, was used. Coriolis was the first to suggest that $\tfrac{1}{2}mv^2$ had a greater physical significance and that it be called vis-viva, but gradually the term kinetic energy has found greater favor. See G. Coriolis, "Calcul de l'effet de machines," Paris, 1829.

[2] See the authors' "Engineering Mechanics," 2d ed., p. 371.

Such an equation can be written for each particle of the system. Summing up these equations, we obtain

$$\sum d\left(\frac{1}{2}mv^2\right) = d\sum \frac{mv^2}{2} = \sum F_e\cos(F_e,v)ds + \sum F_i\cos(F_i,v)ds. \quad (e)$$

This equation states that the change in kinetic energy of the entire system, during a slight change in configuration, is equal to the corresponding work of all external and internal forces together. It must be noted that in general, the work of internal forces does not vanish, since the work of two equal and opposite forces is not zero unless the mutual distance of their points of application remains unchanged.

In our further discussion, we introduce the symbol T for the total kinetic energy of the system. Equation (e) then becomes

$$dT = \Sigma F_e\cos(F_e,v)ds + \Sigma F_i\cos(F_i,v)ds. \quad (108)$$

If the particles are all rigidly connected between themselves, as in the case of a rigid body, the work of internal forces vanishes and we have

$$dT = \Sigma F_e\cos(F_e,v)ds. \quad (109)$$

This equation holds also if the system consists of several rigid bodies connected between themselves and their foundation by frictionless hinges, guides, or other *ideal constraints;* for in such case, the reactions of such constraints do not produce work during motion of the system.[1] As an example of this type of system, we mention here the case of a rigid body that slides along a smooth surface. The reactive force is normal to the direction of motion and does not produce work. Another example arises in the case of a cylinder that rolls, without sliding, along a rough plane. In this case, there is a friction force at the point of contact; but since there is no slipping, it does not produce work and Eq. (109) can be used.

Consider now the case of finite displacement of a system of particles from a configuration A to any other configuration B. Subdividing such displacement into infinitesimal steps and applying Eq. (108) to each step, we obtain, after summation,

$$T_B - T_A = \sum \int_A^B F_e\cos(F_e,v)ds + \sum \int_A^B F_i\cos(F_i,v)ds. \quad (110)$$

This is the general *energy equation* for any system of particles. The left-hand side represents the total change in the kinetic energy of the system between the two configurations A and B; the right-hand side represents the corresponding work of all acting forces, both external and internal.

[1] Such ideal systems will be discussed in more detail in Chap. III.

If the system is such that the work of internal forces vanishes, the energy equation reduces to

$$T_B - T_A = \sum \int_A^B F_e \cos(F_e, v) ds. \tag{111}$$

As a first application of the energy equation, let us consider the motion of a perfectly flexible but inextensible uniform chain along a given smooth curve under the influence of gravity (Fig. 116). The length of chain we denote by $2l$, the weight per unit length by w, and the displacement of its mid-point C by s. All particles evidently have the same velocity v, and hence the total kinetic energy of the system is wlv^2/g. In calculating the work of forces acting on each link, we do not need to consider internal forces, since we assumed an inextensible chain. Likewise, the reactive forces exerted by the smooth surface are normal to the direction of motion at each point, and their work is zero. There remains to be considered, then, only the work of the gravity forces. Corresponding to an infinitesimal positive displacement ds as shown in the figure, this work is evidently equal to the work done in moving the element aa_1, of weight $w\,ds$, up to the position bb_1. Assuming that

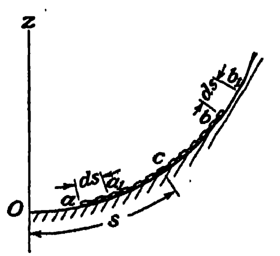


FIG. 116.

$$z = f(s) \tag{f}$$

is the equation of the curve, the required work is

$$-w\,ds[f(s + l) - f(s - l)],$$

and Eq. (109) gives

$$d\left(\frac{wlv^2}{g}\right) = -w\,ds[f(s + l) - f(s - l)]. \tag{g}$$

For a finite displacement of the chain from any position A on the smooth curve to another position B, the total change in kinetic energy is obtained by integration of Eq. (g), which gives

$$\frac{wl}{g}(v_B^2 - v_A^2) = -w \int_A^B [f(s + l) - f(s - l)]ds. \tag{h}$$

From this equation, the velocity of the chain in any position B can be found if its initial velocity in position A is given and the equation of the smooth curve is known.

Assume, for example, that the curve is a *helice* defined by the equation

$$z = \alpha s.$$

Then Eq. (g) becomes

$$d\left(\frac{wlv^2}{g}\right) = -2wl\alpha\, ds$$

or

$$v\, dv = -\alpha g\, ds.$$

Dividing this by dt and observing that $ds/dt = v$, we obtain

$$\frac{dv}{dt} = -\alpha g.$$

This equation is independent of l, and the chain moves like a particle sliding along a smooth inclined plane.

If the smooth curve in Fig. 116 is a *cycloid* defined by the equation

$$z = \frac{s^2}{8R},$$

Eq. (g) becomes

$$d\left(\frac{wlv^2}{g}\right) = -\frac{w}{8R}\, 4ls\, ds,$$

from which

$$\frac{dv}{dt} = \frac{d^2s}{dt^2} = -\frac{gs}{4R}.$$

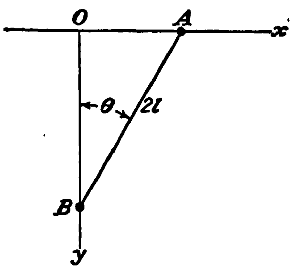


FIG. 117.

We see that in this case, the chain will perform simple harmonic motion having the period

$$\tau = 2\pi\sqrt{\frac{4R}{g}}.$$

As another example, let us consider the motion of two identical particles of mass m joined by a rigid but weightless bar AB of length $2l$ (Fig. 117). The particle A is constrained to move along the horizontal x-axis, while the particle B moves along the vertical y-axis. The configuration of the system is defined by the angle θ as shown. In such case, the displacements of A and B, respectively, are

$$x_a = 2l\sin\theta, \qquad y_b = 2l\cos\theta.$$

The kinetic energy of the system is

$$\frac{m}{2}(\dot{x}_a^2 + \dot{y}_b^2) = 2l^2m\dot{\theta}^2. \tag{i}$$

Since the bar AB is assumed to be absolutely rigid, the work of internal forces is zero. Assuming that the x- and y-axes act as frictionless guides,

the work of reactive forces also vanishes. Hence, we have to consider only the work of the vertical gravity force mg acting on the partical B, and Eq. (109) becomes

$$d(2l^2m\dot{\theta}^2) = mg\,dy_b = -mg2l\sin\theta\,d\theta. \tag{j}$$

After differentiation, this becomes

$$\frac{d^2\theta}{dt^2} = -\frac{g}{l}\sin\theta,$$

which has the form of the equation of motion of a mathematical pendulum of length l.

In calculating the kinetic energy of a system of particles, it is sometimes advantageous to resolve the velocity of each particle into two components: (1) the velocity of translation with the mass-center of the system and (2) the velocity of relative motion with respect to the mass-center. With reference to fixed orthogonal axes x,y,z, we define the coordinates of the mass-center C of a given system of particles by the equations (see page 107)

$$x_c = \frac{\Sigma mx}{\Sigma m}, \qquad y_c = \frac{\Sigma my}{\Sigma m}, \qquad z_c = \frac{\Sigma mz}{\Sigma m}. \tag{k}$$

Taking the mass-center C as the origin of moving coordinate axes ξ, η, ζ parallel, respectively, to x,y,z, the absolute coordinates of any particle of the system will be

$$x = x_c + \xi, \qquad y = y_c + \eta, \qquad z = z_c + \zeta,$$

and the projections of its velocity will be

$$\dot{x} = \dot{x}_c + \dot{\xi}, \qquad \dot{y} = \dot{y}_c + \dot{\eta}, \qquad \dot{z} = \dot{z}_c + \dot{\zeta}.$$

Substituting these velocity components into expression (a) for kinetic energy and summing up for all particles of the system, we obtain

$$T = \tfrac{1}{2}\Sigma m(\dot{x}_c^2 + \dot{y}_c^2 + \dot{z}_c^2) + \tfrac{1}{2}\Sigma m(\dot{\xi}^2 + \dot{\eta}^2 + \dot{\zeta}^2) + \Sigma m(\dot{x}_c\dot{\xi} + \dot{y}_c\dot{\eta} + \dot{z}_c\dot{\zeta}). \tag{l}$$

From the definition of mass-center, it follows that

$$\Sigma m\xi = \Sigma m\eta = \Sigma m\zeta = 0,$$
$$\Sigma m\dot{\xi} = \Sigma m\dot{\eta} = \Sigma m\dot{\zeta} = 0.$$

Hence the last term in expression (l) vanishes; and with the notations

$$v_c^2 = \dot{x}_c^2 + \dot{y}_c^2 + \dot{z}_c^2, \qquad M = \Sigma m,$$

we obtain

$$T = \tfrac{1}{2}Mv_c^2 + \tfrac{1}{2}\Sigma m(\dot{\xi}^2 + \dot{\eta}^2 + \dot{\zeta}^2). \tag{112}$$

From this expression, we see that the total kinetic energy of a system of particles can be considered as consisting of two parts: (1) the kinetic energy of translation, calculated as if the total mass M were concentrated at the mass-center and moving with the velocity of that point, and (2) the kinetic energy of relative motion, calculated as if the axes ξ, η, ζ were fixed.

In the particular case of plane motion of a rigid body (Fig. 118), the only possibility of relative motion with respect to the moving ξ-, η-, ζ-axes through the center of gravity C of the body is rotation around the ζ-axis. In such case, the motion of the system is completely defined by the velocity v_c of the center of gravity C, and the angular velocity ω and expression (112) for the total kinetic energy can be written in a simpler form. The two components of velocity for any particle of mass m at the distance ρ from the ζ-axis are v_c and $\rho\omega$ as shown, and the expression for total kinetic energy becomes

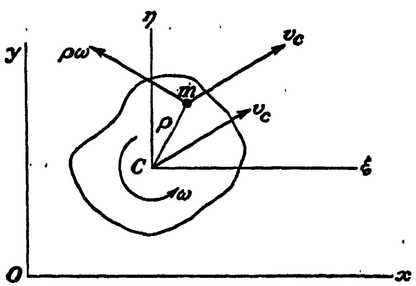


FIG. 118.

$$T = \frac{1}{2}\sum mv_c^2 + \frac{1}{2}\sum m(\rho\omega)^2 = \frac{v_c^2}{2}\sum m + \frac{\omega^2}{2}\sum \rho^2 m$$
$$= \frac{1}{2}Mv_c^2 + \frac{1}{2}I_c\omega^2, \quad (m)$$

where $I_c = \Sigma\rho^2 m$ is the moment of inertia of the body with respect to the ζ-axis. Denoting by i_c, the radius of gyration of the body with respect to the centroidal ζ-axis, expression (m) can be put in the more convenient form

$$T = \tfrac{1}{2}M(v_c^2 + i_c^2\omega^2). \quad (113)$$

In the general case of motion of a rigid body, we consider the motion at any instant as a combination of translation with velocity v_c and rotation with angular velocity ω about an instantaneous axis through the center of gravity C. Then the total kinetic energy is

$$T = \tfrac{1}{2}Mv_c^2 + \tfrac{1}{2}I\omega^2, \quad (n)$$

where I is the moment of inertia with respect to the instantaneous axis of rotation through C. During motion, the orientation of this axis is changing, and the angular velocity ω must be resolved into components ω_1, ω_2, and ω_3, corresponding to the three principal central axes of the

body. In such case, ω_1/ω, ω_2/ω, ω_3/ω represent the cosines of the angles that the instantaneous axis makes with the principal axes of the body, and we have[1]

$$I = I_1 \frac{\omega_1^2}{\omega^2} + I_2 \frac{\omega_2^2}{\omega^2} + I_3 \frac{\omega_3^2}{\omega^2}.$$

Substituting this into expression (n), we have

$$T = \tfrac{1}{2}Mv_c^2 + \tfrac{1}{2}(I_1\omega_1^2 + I_2\omega_2^2 + I_3\omega_3^2). \tag{114}$$

To show the application of this expression, let us consider the kinetic energy of a circular disk of radius a that rolls, without slipping, along a horizontal circular track of radius b (Fig. 119). The velocity of the center of gravity C we denote by v_c. Then, without slip, the components of the angular velocity of the disk with respect to its principal axes are

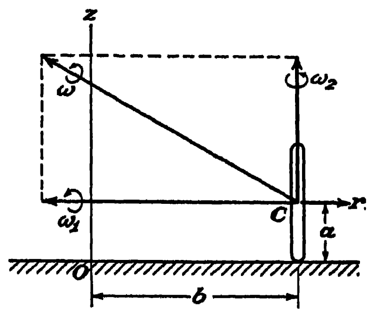


FIG. 119.

$$\omega_1 = \frac{v_c}{a}, \qquad \omega_2 = \frac{v_c}{b}, \qquad \omega_3 = 0,$$

as shown in the figure. Assuming a thin disk, the corresponding moments of inertia are

$$I_1 = \tfrac{1}{2}Ma^2, \qquad I_2 = \tfrac{1}{4}Ma^2, \qquad I_3 = \tfrac{1}{4}Ma^2,$$

and Eq. (114) gives

$$T = \frac{1}{2}Mv_c^2\left(\frac{3}{2} + \frac{a^2}{4b^2}\right).$$

Having the kinetic energy of a system of particles as represented by Eq. (112), we proceed now to derive the energy equation for relative motion of a system of particles with respect to their mass-center. Using previous notations and considering a small change in configuration of the system, the displacement of any particle will be

$$dx = dx_c + d\xi, \qquad dy = dy_c + d\eta, \qquad dz = dz_c + d\zeta.$$

Substituting these expressions into Eq. (108) and using expression (b) for work, we obtain

$$\begin{aligned} dT = {} & \Sigma(X_e dx_c + Y_e dy_c + Z_e dz_c) + \Sigma(X_e d\xi + Y_e d\eta + Z_e d\zeta) \\ & + \Sigma(X_i dx_c + Y_i dy_c + Z_i dz_c) + \Sigma(X_i d\xi + Y_i d\eta + Z_i d\zeta). \end{aligned} \tag{o}$$

To simplify this equation, we observe that the quantities dx_c, dy_c, dz_c are

[1] *Ibid.*, p. 513.

to be treated as constants during summation and that

$$\Sigma X_i = \Sigma Y_i = \Sigma Z_i = 0,$$

since the internal forces appear as pairs of equal and opposite forces. Thus the third term on the right-hand side of Eq. (*o*) vanishes. Observing further that the mass-center of the system moves as a particle of mass $M = \Sigma m$ to which a force equal to the geometric sum of all external forces acting on the system is applied, we conclude that

$$d\left(M\frac{v_c^2}{2}\right) = \sum (X_e dx_c + Y_e dy_c + Z_e dz_c).$$

Using this result in place of the first term on the right-hand side of Eq. (*o*) and substituting for T its expression (112), we obtain

$$d[\tfrac{1}{2}\Sigma m(\dot{\xi}^2 + \dot{\eta}^2 + \dot{\zeta}^2)] = \Sigma(X_e d\xi + Y_e d\eta + Z_e d\zeta) + \Sigma(X_i d\xi + Y_i d\eta + Z_i d\zeta). \quad (115)$$

This shows that the energy equation applies also to the relative motion of a system of particles with respect to coordinate axes ξ, η, ζ of invariable direction and origin C moving with the mass-center of the system. That is, the change in kinetic energy of relative motion alone is equal to the work of all acting forces on the corresponding relative displacements only. Applying Eq. (115) to our solar system, we conclude that changes in the relative motions of sun and planets which we are able to observe are entirely independent of how the mass-center of the system as a whole may be moving through space.

Like the momentum principles, the energy equation also finds a wide application in the field of hydrodynamics. As an example, let us consider the steady motion of an incompressible frictionless fluid. In such case, we do not try to retain the identity of individual fluid particles but visualize the motion in terms of a system of *streamlines*, tangents to which give the directions of velocity at all points in the field. Drawing such streamlines through all points of any small closed curve in the fluid field, we obtain a *stream tube*. In the case of steady flow, the streamlines are invariable with time and the fluid that moves through the chosen stream tube behaves exactly as if the tube were real. Referring to Fig. 120, let AB represent an arbitrary portion of such a tube. In our further discussion, we confine our attention to the system of fluid particles initially contained between the cross sections A and B of this tube. After a lapse of time dt, the same fluid mass will occupy the volume A_1B_1. The external forces acting on this system of particles are its own weight and the pressure forces distributed over its surface. In calculating the work of these forces on the indicated infinitesimal displacement, we begin

with the work of gravity forces and observe that this will be represented by the work done in moving the fluid AA_1 at the elevation z_1 down to BB_1 at the elevation z_2. Denoting by Q the volume of fluid passing any given cross section in unit time, by a_1 and v_1 the cross-sectional area and

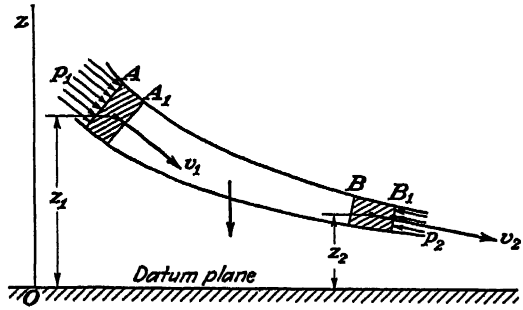


FIG. 120.

velocity at A, and by a_2 and v_2 the same quantities at B, we have, for the shaded volumes,

$$a_1v_1dt = a_2v_2dt = Q\,dt.$$

Multiplying by w, the specific weight of the fluid, we obtain the weight of fluid AA_1 or BB_1. When the fluid moves from configuration AB to configuration A_1B_1, the total work of gravity forces is then

$$Qw\,dt(z_1 - z_2).$$

In calculating the work of pressure forces during motion from AB to A_1B_1, we denote by p_1 and p_2 the pressures at cross sections A and B, respectively. Then the works of these pressure forces are, respectively,

$$p_1a_1v_1dt = p_1Q\,dt \qquad \text{and} \qquad -p_2a_2v_2dt = -p_2Q\,dt.$$

The pressures on the lateral surface of the tube are all normal to the direction of motion, and their work is zero. Thus the total work of external forces acting on the considered system of particles is

$$Qw(z_1 - z_2)dt + Q(p_1 - p_2)dt. \tag{p}$$

Since we are dealing with an incompressible fluid, the distances between particles of which do not change, the work of internal forces vanishes.

In calculating the change in kinetic energy of our system of particles, we observe that during the assumed displacement, the velocities of particles within the region A_1B do not change. Hence we will obtain the total change in kinetic energy simply as the difference between the energies of the two shaded elements in the figure. Thus

$$dT = \frac{Qw}{g}\frac{v_2^2}{2}dt - \frac{Qw}{g}\frac{v_1^2}{2}dt. \tag{q}$$

Equating expressions (p) and (q), we obtain the energy equation

$$\frac{Qw}{g}\frac{v_2^2}{2}dt - \frac{Qw}{g}\frac{v_1^2}{2}dt = Qw(z_1 - z_2)dt + Qp_1dt - Qp_2dt,$$

which is readily reduced to[1]

$$\frac{v_1^2}{2g} + \frac{p_1}{w} + z_1 = \frac{v_2^2}{2g} + \frac{p_2}{w} + z_2. \tag{116}$$

In using this equation, we must remember that it applies only to the special case of steady flow of an ideal incompressible fluid.

Since the cross sections A and B of the stream tube in Fig. 120 were chosen arbitrarily, we conclude, in general, that for any cross section, the velocity v, the pressure p, and the elevation z must be so related as to satisfy the equation

$$\frac{v^2}{2g} + \frac{p}{w} + z = \text{const.} \tag{117}$$

This is a statement of the law of conservation of energy as it applies to the steady flow of an ideal incompressible fluid, for the sum of the three terms can be shown to represent the total energy per unit weight of the flowing fluid.* To show this, we start with 1 lb. of fluid at rest on the datum plane and therefore having zero energy. First, we raise it up to the elevation z, which evidently requires the work $1 \cdot z$. Next, we must force it into the stream tube against the pressure p, and the work required to do this is the product of the pressure p and the volume $(1/w)$ of the 1 lb. of fluid, *i.e.*, $p(1/w)$. Finally, to bring it from rest up to the velocity v requires the work $(1/g)(v^2/2)$, and we have total energy acquired equal to total work done equal to

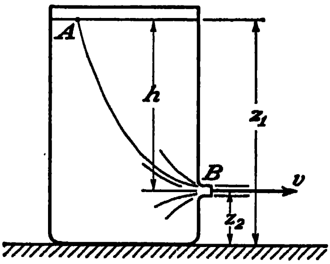


FIG. 121.

$$z + \frac{p}{w} + \frac{v^2}{2g} = \text{const.}$$

As an application of Bernoulli's equation (116), let us consider efflux of water from a vessel in the side of which is a small opening B (Fig. 121). If we trace back the streamlines from B, we find that they lead to the free surface A. Assuming that the water surface at A is large compared with the opening at B, we can neglect v_A^2 at A in comparison

[1] This is the celebrated Bernoulli theorem, after Daniel Bernoulli, 1738.

* If we multiply each term in Eq. (116) by the gravitational constant g, we obtain total energy per unit mass of fluid.

with v^2 at B. Also, p_A and p_B are both atmospheric pressures, and Eq. (116) becomes

$$z_1 - z_2 = h = \frac{v^2}{2g},$$

from which[1]

$$v = \sqrt{2gh}. \tag{r}$$

We see that the velocity of a particle of water in the free jet at B is the same as if the particle had fallen freely through the height h.

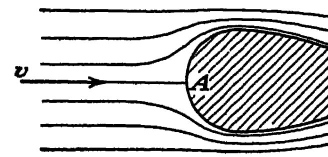


FIG. 122.

As another example, consider the pressure that a uniformly flowing stream exerts on the nose of an obstruction (Fig. 122). The fluid will be dammed up immediately in front of the obstacle, and at point A, called the *stagnation point*, it comes completely to rest. Denoting the pressure here by p_s and that at the same depth in the undisturbed stream by p_0, Eq. (116) gives

$$\frac{v^2}{2g} + \frac{p_0}{w} = \frac{p_s}{w},$$

from which

$$p_s = p_0 + \frac{wv^2}{2g}. \tag{s}$$

We see that the total pressure p_s at A consists of two parts: (1) the static pressure p_0 and (2) the dynamic pressure $wv^2/2g$. These concepts are widely used in aerodynamical studies.

PROBLEMS

81. A uniform flexible chain of length l and weight wl rests on a smooth horizontal table with an initial overhang x_0 (Fig. 123). Using the energy equation, derive the general expression for its velocity after release. *Ans.* $v = \sqrt{(g/l)(x^2 - x_0^2)}$.

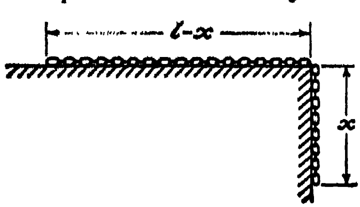


FIG. 123.

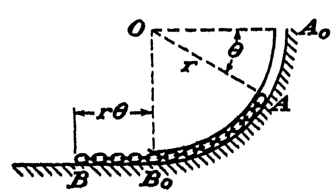


FIG. 124.

82. A perfectly smooth tube in the form of a quarter circle of radius r contains a chain of length $l = \pi r/2$ and weight $w\pi r/2$ and stands in a vertical plane as shown in Fig. 124. If the chain is released from rest inside the tube and slides out along a smooth horizontal plane, find its velocity v for any position as defined by the angle θ.
Ans. $v = 2\sqrt{(gr/\pi)(\theta + \cos\theta - 1)}$.

[1] Equation (r) represents the so-called *Torricelli theorem*.

83. A perfectly smooth semicircular tube of radius r standing in a vertical plane contains a chain of length πr and weight $w\pi r$ as shown in Fig. 125. Due to a slight disturbance, the chain begins to slide out of the open end of the tube as shown. Find the velocity v of the chain for any value of θ.

Ans. $v = \sqrt{(gr/\pi)(2 - 2\cos\theta + \theta^2)}$.

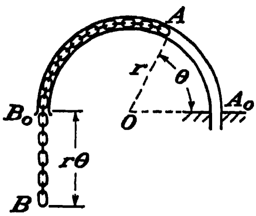


FIG. 125.

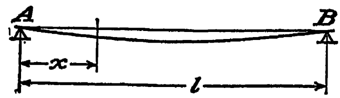


FIG. 126.

84. Calculate the kinetic energy of the vibrating beam in Fig. 126 if the vertical velocity at any cross section is $v_0 \sin(\pi x/l)$ and the weight of the beam is wl.

Ans. $T = (wl/4g)v_0^2$.

85. A right circular cylindrical roller of radius r and weight W rolls, without slipping, in a cylindrical seat of radius R (Fig. 127). The axes of the two cylinders are parallel and horizontal, and the motion is completely defined by the angular velocity $\dot{\phi}$ of the line OC. Find the total kinetic energy of the roller.

Ans. $T = \frac{3}{4}(W/g)(R - r)^2\dot{\phi}^2$.

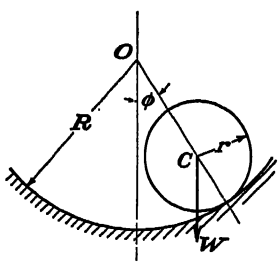


FIG. 127.

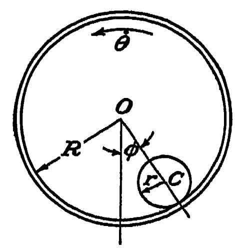


FIG. 128.

86. A hollow right circular shell of radius R and weight W rotates about its horizontal geometric axis OO with angular velocity $\dot{\theta}$. Inside this shell is a solid right circular cylinder of radius r and weight w as shown in Fig. 128. The position of the cylinder inside the shell is defined by the angle ϕ as shown. Write the general expression for the kinetic energy of the system if there is no slip.

$$Ans.\quad T = \frac{W + \frac{1}{2}w}{2g} R^2\dot{\theta}^2 + \frac{3}{4}\frac{w}{g}(R - r)^2\dot{\phi}^2 - \frac{w}{2g}R(R - r)\dot{\phi}\dot{\theta}.$$

87. A homogeneous right circular cone of weight W, slant height l, and vertex angle 2α rolls, without slipping, on a rough horizontal plane as shown in Fig. 129. Its motion is completely defined by the angular velocity $\dot{\theta}$ around the vertical z-axis through the vertex O. Develop the general expression for the total kinetic energy of the cone.

Ans. $T = \frac{3}{20}(W/g)\, l^2\cos^2\alpha(3\cos^2\alpha + \frac{1}{2}\sin^2\alpha)\dot{\theta}^2$.

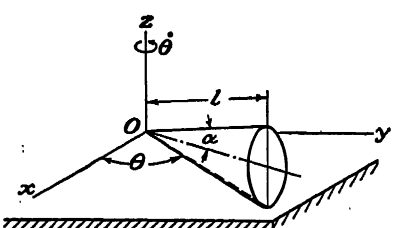


FIG. 129.

88. The governor shown in Fig. 130 consists of two flyballs each of mass m_1 and a sliding weight of mass m_2. The system rotates about the fixed vertical axis with

angular velocity ω. The moment of inertia of the T-shaft is I. Neglecting masses of the arms of length l, find the total kinetic energy of the system.

Ans. $T = \frac{1}{2}I\omega^2 + m_1[(a + l \sin \phi)^2\omega^2 + l^2\dot{\phi}^2] + \frac{1}{2}m_2(4l^2 \sin^2\phi)\dot{\phi}^2$.

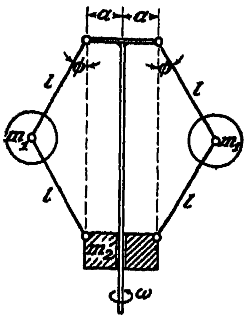


FIG. 130.

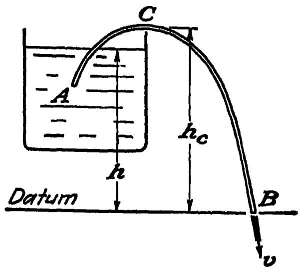


FIG. 131.

89. Water is siphoned out of an open tank through the bent tube ACB shown in Fig. 131. Neglecting friction, find the velocity v of the flow. What gauge pressure p_c exists inside the tube at C? *Ans.* $v = \sqrt{2gh}$; $p_c = -wh_c$.

90. A fire hose (inside diameter 3 in.) has a nozzle at its end and delivers 250 g.p.m. of water in a 1-in.-diameter jet. What gauge pressure p exists in the hose at the base of the nozzle? *Ans.* $p = 69.2$ p.s.i.

20. The Law of Conservation of Energy.—The energy equation (110) developed in the preceding article can be applied to a system of particles acted upon by any kind of forces. Frequently, however, we have to deal with systems involving only such forces the work of which, during displacement from one position A to another position B, depends only on the initial and final configurations of the system, and is independent of the actual paths that the various particles describe during the displacement. Such forces are said to have a *potential.*

The simplest case of force having a potential is the force of gravity. Consider, for example, a particle of weight w initially at A with elevation z_1 above an arbitrary horizontal reference plane. During displacement of the particle to any other position B with elevation z_2, the gravity force w, opposing this displacement, produces the work $-w(z_2 - z_1)$. This work is independent of the actual path of the particle between A and B and depends only on the initial and final positions. If the particle moves from B back to A, the gravity force produces the work $+w(z_2 - z_1)$. In general, the work wz that the gravity force can produce when the particle comes from any elevation z back to the reference plane is called the *potential energy* of the particle at the elevation z and will be denoted by V. With this notation, we see that the work done by the gravity force when the particle is displaced from A to B is

$$-w(z_2 - z_1) = -(V_2 - V_1);$$

i.e., it is equal to the change in potential energy with reversed sign.

If we have a system of particles of weights $w_1, w_2, \ldots w_n$, at the elevations $z_1, z_2, \ldots z_n$ above the reference plane, the corresponding potential energy of the system is

$$V = w_1 z_1 + w_2 z_2 + \cdots + w_n z_n = \Sigma(wz).$$

Using expression (k), page 157, for the coordinates of the mass-center of a system, we can represent this potential energy in the simpler form

$$V = \Sigma(wz) = Wz_c, \tag{a}$$

where W is the total weight of all particles and z_c is the elevation of their mass-center.

As an example of internal forces having a potential, let us consider the case of two particles connected by a spring with the characteristic k.* Let any configuration of the system in this case be defined by an extension x of the spring; then the force acting on each particle is kx. These forces produce work only if the distance between the particles changes. Giving a small increase dx to the extension x in the spring, we see that the forces kx oppose this increase and produce work of the amount $-kx\,dx$. Summing up such increments from any initial extension x_1 to a final extension x_2, we obtain for the total work

$$-\int_{x_1}^{x_2} kx\,dx = -\left(\frac{kx_2^2}{2} - \frac{kx_1^2}{2}\right).$$

Again, we see that this work is independent of the actual paths of the two particles during the displacement and depends only on the initial and final configurations. The work done by the internal forces kx when the spring is allowed to shorten from any extension x to the unstrained configuration ($x = 0$) represents the potential energy of the system in the strained configuration. Thus, in this case,

$$V = \frac{kx^2}{2}. \tag{b}$$

Using this notation in the above expression for work during any change in configuration from initial extension x_1 to final extension x_2, we have

$$-\int_{x_1}^{x_2} kx\,dx = -(V_2 - V_1),$$

and, again, the work is represented by the change in potential energy with reversed sign.

Let us consider next the case of two particles of masses m_1 and m_2 that are separated by the distance r and mutually attract each other with

* The force required to produce unit extension of the spring.

the force $\mu m_1 m_2/r^2$. If the distance r between these two particles is given a slight increase dr, the forces of attraction oppose this separation and produce the work $-dr(\mu m_1 m_2/r^2)$. Summing up such increments of work during a change in configuration of the system from initial mutual distance r_1 to final mutual distance r_2, we obtain for the total work

$$-\mu m_1 m_2 \int_{r_1}^{r_2} \frac{dr}{r^2} = \mu m_1 m_2 \left(\frac{1}{r_2} - \frac{1}{r_1}\right).$$

Again, this work is seen to depend only on the initial and final configurations of the system, and we have forces with a potential. Taking $r_0 = \infty$ as the reference configuration in this case and proceeding as in previous cases, we find, for the potential energy in any other configuration, defined by the mutual distance r,

$$V = -\frac{\mu}{r} m_1 m_2, \tag{c}$$

and the above expression for work becomes

$$-\mu m_1 m_2 \int_{r_1}^{r_2} \frac{dr}{r^2} = -(V_2 - V_1).$$

If to a system of two particles m_1 and m_2 at the mutual distance r_{12} we bring a third particle m_3 from infinity, the increase in potential energy of the system is evidently

$$\Delta V = -\mu m_3 \left(\frac{m_1}{r_{13}} + \frac{m_2}{r_{23}}\right).$$

Thus we conclude that the total potential energy of a system of n particles at the mutual distances r_{ij} will be

$$V = -\mu \sum_{i=1}^{i=n-1} \left(m_i \sum_{j=i+1}^{j=n} \frac{m_j}{r_{ij}}\right) \cdot \tag{c'}$$

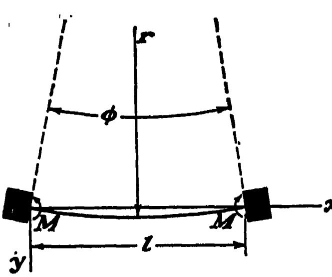


FIG. 132.

We consider, now, two bodies joined together by an inextensible bar of length l and flexural rigidity EI (Fig. 132). Let any configuration of the system be defined by the relative angle of rotation ϕ of the two bodies. Then the moments M exerted on the two bodies by the ends of the bent bar will be of magnitude $k\phi$, where $k = EI/l$ is the flexural spring characteristic for the bar. During an increase $d\phi$ in the angle ϕ, these moments produce the work $-k\phi\, d\phi$. Summing such increments for any change in configuration from ϕ_1 to ϕ_2, the total work done is

$$-\int_{\phi_1}^{\phi_2} k\phi\, d\phi = -\left(\frac{k\phi_2^2}{2} - \frac{k\phi_1^2}{2}\right).$$

Proceeding as in previous cases and allowing the system to return from any strained configuration, defined by the relative angle of rotation ϕ, to the unstrained configuration ($\phi = 0$), we find that its potential energy is

$$V = \frac{k\phi^2}{2} = \frac{kl^2}{2r^2} = \frac{EIl}{2r^2}, \tag{d}$$

where r is the radius of curvature of the bent bar.

In the more general case of bending of a beam in one of its principal planes, the radius of curvature may vary along the length of the beam. In such case, we can use the above formula (d) only for one element of infinitesimal length dx. Then the total potential energy of the distorted beam is

$$V = \int_0^l \frac{EI}{2r^2}\, dx \approx \frac{EI}{2}\int_0^l \left(\frac{d^2y}{dx^2}\right)^2 dx. \tag{d'}$$

Sometimes we have to calculate the potential energy of pressure p uniformly distributed over the surface of a body of variable volume. Let dS be an element of surface of the body and dn its displacement in the outward normal direction during a small change dv in the volume. Then the work done by the pressure force $p\,dS$ acting on that element is $-p\,dS\,dn$, and the corresponding work over the entire surface S is

$$-\int_S p\, dS\, dn = -p\, dv.$$

If the pressure p is a function of the volume and we consider a change in configuration from volume v_1 to volume v_2, the total work is

$$-\int_{v_1}^{v_2} p\, dv.$$

Allowing the body to return from any volume v to an arbitrarily chosen reference volume v_0, the work done is

$$-\int_v^{v_0} p\, dv;$$

and hence for potential energy at the volume v, we have

$$V = \int_{v_0}^{v} p\, dv. \tag{e}$$

In the particular case where the pressure p remains constant, this becomes

$$V = p(v - v_0). \tag{e'}$$

In all cases like those discussed above, where the acting forces (either external or internal) have a potential, we have seen that their work during any displacement of the system from one configuration A to another configuration B can be expressed as the corresponding change in potential energy with reversed sign. That is,

$$\sum \int_A^B F \cos (F,v) ds = \sum \int_0^B F \cos(F,v) ds - \sum \int_0^A F \cos(F,v) ds$$
$$= -(V_B - V_A). \qquad (f)$$

Substituting this expression on the right-hand side of the energy equation (110), we obtain

$$T_B - T_A = -(V_B - V_A),$$

from which

$$T_B + V_B = T_A + V_A. \qquad (118)$$

Since the configurations A and B of the system are entirely arbitrary, we conclude that in the case of forces having a potential, the sum of the kinetic and potential energies of the system remains constant during its motion. This is the *law of conservation of energy*. Systems for which this law holds are called *conservative systems*. If during motion of a conservative system the kinetic energy T increases by some amount, then the potential energy V must decrease by the same amount and vice versa. If after some motion the system passes again through its initial configuration A, the potential energy V, depending only on configuration, acquires its initial value; hence the kinetic energy also must have its initial value.

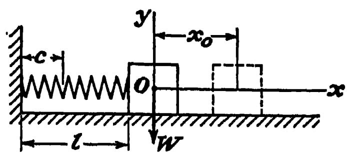


FIG. 133.

As an application of the law of conservation of energy, let us consider the free vibrations of a block of weight W attached to a spring of characteristic k and sliding on a smooth horizontal track as shown in Fig. 133. Measuring displacements from the position of the block corresponding to the unstressed configuration of the spring, we assume, as initial conditions, $x = x_0$, $\dot{x} = 0$, when $t = 0$. For any other position, during vibration, the forces acting on the block are the spring force $-kx$, the gravity force W, and the reaction exerted by the plane. Since the last two forces are always balanced, we have to consider only the spring force, the potential energy of which is $kx^2/2$. For the initial configuration, the potential energy is $kx_0^2/2$ and the kinetic energy is zero. Hence Eq. (118) gives

$$\frac{W}{g}\frac{\dot{x}^2}{2} + \frac{kx^2}{2} = \frac{kx_0^2}{2}. \qquad (g)$$

We take the solution of this equation in the form

$$x = x_0 \cos pt, \tag{h}$$

which satisfies the initial conditions and where p is an undetermined constant. To find the value of p, we substitute the assumed solution (h) into Eq. (g), which gives

$$\frac{W}{2g} x_0^2 p^2 \sin^2 pt + \frac{kx_0^2}{2} \cos^2 pt = \frac{kx_0^2}{2}.$$

This equation is satisfied if $Wp^2/g = k$, which yields $p = \sqrt{kg/W}$, and the period of vibration is

$$\tau = \frac{2\pi}{p} = 2\pi \sqrt{\frac{W}{kg}}. \tag{i}$$

The above approach to the problem of free vibrations of a system has certain advantages over working directly with the differential equation of motion as we did in the preceding chapter, since it can be used with equal success in more complicated cases. Suppose, for example, that we wish to investigate the effect of the distributed mass of the spring on the period of free vibration of the system in Fig. 133. If the spring were truly without mass, as we tacitly assumed in the preceding discussion, its configuration during vibration would be the same as under purely static stretching. That is, if x is the displacement of the block at any instant, the displacement of any element of the spring at the natural distance c from the fixed end will be cx/l. If the mass of the spring is not neglected, the problem of free vibrations of the system becomes much more complicated, since various modes of vibration may occur. But if we are interested in the fundamental mode only, an approximate value for the period τ can be obtained by using the law of conservation of energy. In making such an approximate calculation, we assume that during vibration the configuration of the spring is the same as if it had no mass. Then the potential energy of the system for any displacement x of the block will be the same as before, and the kinetic energy, including that of the moving mass elements of the spring, can easily be calculated by observing that the velocity of any such element is $c\dot{x}/l$, where $\dot{x}$ is the velocity of the attached block. Denoting by w the weight per unit length of the spring and by l its unstrained length, we obtain, for its kinetic energy, the expression

$$T_s = \int_0^l \frac{w}{2g} \frac{c^2\dot{x}^2}{l^2} dc = \frac{wl}{3}\left(\frac{\dot{x}^2}{2g}\right).$$

Adding this to the kinetic energy of the block and using Eq. (118), we obtain

$$\left(W + \frac{1}{3}wl\right)\frac{\dot{x}^2}{2g} + \frac{kx^2}{2} = \frac{kx_0^2}{2} \tag{j}$$

instead of Eq. (*g*) above and, for the period of vibration,

$$\tau = 2\pi\sqrt{\frac{W + \frac{1}{3}wl}{kg}} \tag{k}$$

instead of Eq. (*i*). We see that an approximate correction for the effect of the mass of the spring on the natural period of vibration of the system will be obtained by adding one-third of the spring mass to the attached mass and then proceeding as for a massless spring.

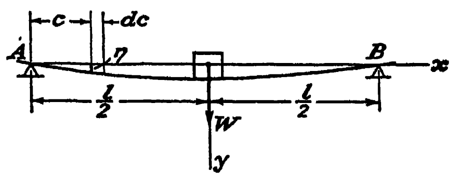


FIG. 134.

As a second application of the law of conservation of energy, let us consider free vertical vibrations of a block of weight W supported at the middle of a simply supported beam AB (Fig. 134). Neglecting, first, the weight of the beam and measuring vertical displacements of the block, during vibration, from its equilibrium position, we conclude that the kinetic energy of the system at any instant is

$$T = W\left(\frac{\dot{y}^2}{2g}\right).$$

In calculating the potential energy, we have to consider the gravity force W and the elastic reaction R that the beam exerts on the block. The potential energy of the gravity force for any displacement y of the block from its equilibrium position is $-Wy$. The elastic reaction R is proportional to the deflection of the beam from the unstrained configuration. Denoting the spring characteristic of the beam by k* and the static deflection W/k by δ_{st}, we have

$$R = -k(\delta_{st} + y),$$

and the potential energy of this force is

$$-\int_y^0 k(\delta_{st} + y)dy = k\delta_{st}y + \frac{1}{2}ky^2 = Wy + \frac{ky^2}{2}.$$

* For a simply supported beam of flexural rigidity EI loaded at the middle, $k = 48EI/l^3$.

Adding to this the potential $-Wy$ of the gravity force and using Eq. (118), we obtain

$$\frac{W\dot{y}^2}{2g} + \frac{ky^2}{2} = \frac{ky_0^2}{2}, \tag{l}$$

where y_0 is the amplitude of vibration. This is the same equation as Eq. (g) above, and hence the period of vibration is

$$\tau = 2\pi\sqrt{\frac{W}{kg}} = 2\pi\sqrt{\frac{\delta_{st}}{g}}. \tag{m}$$

Now to make an approximate correction for the effect of the distributed mass of the beam on this period of vibration, we assume, as before, that during vibration, the shape of the beam is always similar to a static deflection curve for a load P at the middle. Then for any displacement y of the block from its middle position, the corresponding displacement η of an element dc of the beam, distance c from the left end, will be[1]

$$\eta = y\frac{3cl^2 - 4c^3}{l^3}.$$

The velocities $\dot{\eta}$ and $\dot{y}$ will be in the same ratio, and we obtain, for the kinetic energy of the beam

$$T_b = 2\int_0^{\frac{l}{2}} \dot{\eta}^2 \frac{w\,dc}{2g} = 2\int_0^{\frac{l}{2}} \dot{y}^2\left(\frac{3cl^2 - 4c^3}{l^3}\right)^2 \frac{w\,dc}{2g} = \frac{17}{35}\frac{wl}{2g}\dot{y}^2,$$

where w is the weight per unit length of the beam. Adding this to the kinetic energy of the block and using Eq. (118), we obtain

$$\frac{ky^2}{2} + \frac{\dot{y}^2}{2g}\left(W + \frac{17}{35}wl\right) = \frac{ky_0^2}{2} \tag{n}$$

instead of Eq. (l) above. We see now that to correct for the effect of the mass of the beam on the period of free vibration, we need only to add to the weight W of the block $\frac{17}{35}$ of the weight of the beam and then use formula (m) as for a weightless beam.

As a last example, let us calculate the period of lateral vibrations of the beam in Fig. 134 without the block W, assuming that during motion, the configuration of the beam is defined by the equation

$$\eta = y\sin\left(\frac{\pi x}{l}\right),$$

where y as before, is the displacement of the mid-point of the beam from

[1] See S. Timoshenko "Strength of Materials," 2d ed., pt. I, p. 156.

the equilibrium configuration. Differentiating this expression with respect to time, we find that the velocity of any element of the beam is $\dot{\eta} = \dot{y} \sin (\pi x/l)$, and the expression for the total kinetic energy of the system becomes

$$T = \frac{w}{g}\frac{\dot{y}^2}{2}\int_0^l \sin^2\left(\frac{\pi x}{l}\right)dx = \frac{wl}{2}\frac{\dot{y}^2}{2g}, \tag{o}$$

where wl is the total weight of the beam. Using Eq. (d'), page 168, the potential energy of the beam is

$$V = \frac{EI}{2}\int_0^l \left(\frac{d^2\eta}{dx^2}\right)^2 dx = \frac{\pi^4 EI}{4l^3}\, y^2, \tag{p}$$

with respect to the equilibrium configuration. Assuming

$$y = y_0 \cos pt,$$

we find that when the beam is in an extreme configuration, the potential energy is $\pi^4 EI y_0^2/4l^3$ and, of course, the kinetic energy is zero. Similarly, when the beam passes through its equilibrium configuration, its kinetic energy is $wlp^2y_0^2/4g$ and its potential energy is zero, since we measure displacements η from the equilibrium configuration. The law of conservation of energy now gives

$$\frac{\pi^4 EI}{4l^3}\, y_0^2 = \frac{p^2 wl}{4g}\, y_0^2,$$

from which

$$p^2 = \frac{\pi^4 EIg}{wl^4},$$

and the period of vibration is

$$\tau = \frac{2\pi}{p} = \frac{2l^2}{\pi}\sqrt{\frac{w}{EIg}}. \tag{q}$$

PROBLEMS

91. What portion of the weight wl of the uniform cantilever beam in Fig. 135 should be added to the weight W at the end to correct formula (m) approximately for the period of free lateral vibration. Assume that during vibration, the beam maintains a configuration similar to a static deflection curve for a load at the free end.

Ans. $33wl/140$.

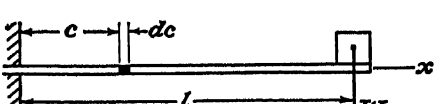


FIG. 135.

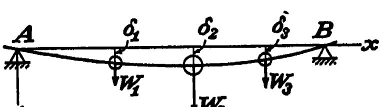


FIG. 136.

92. Referring to Fig. 136, prove that if δ_1, δ_2, δ_3 are the static deflections of a simply supported beam AB under the loads W_1, W_2, W_3, then the approximate expres-

sion for the period of the fundamental mode of vibration of the system is

$$\tau = 2\pi\sqrt{\frac{W_1\delta_1^2 + W_2\delta_2^2 + W_3\delta_3^2}{g(W_1\delta_1 + W_2\delta_2 + W_3\delta_3}}.$$

Neglect the mass of the beam in this case.

93. Find the period of oscillation of the roller in Prob. 85, assuming that it rolls without slipping and that the angular displacement ϕ is always small.

Ans. $\tau = 2\pi\sqrt{\dfrac{3(R-r)}{2g}}.$

94. Assuming that the solid cone described in Prob. 87 is placed on a rough plane inclined to the horizontal by the angle β and that it performs small rolling oscillations about its equilibrium position without slipping, find its period τ.

Ans. $\tau = 2\pi\sqrt{\dfrac{2l}{5g}\left(\dfrac{3\cos^2\alpha + \frac{1}{4}\sin^2\alpha}{\sin\beta}\right)}.$

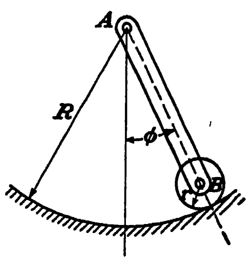


FIG. 137.

95. A slender prismatic bar AB of weight W_1 is hinged at A and swings in its vertical plane as a physical pendulum (Fig. 137). At the free end B of this bar is a roller of weight W_2 and radius r that rolls without slipping along a circular track of radius R with center at A as shown. Find the natural period τ of the system for small oscillations in the plane of the figure.

Ans. $2\pi\sqrt{\dfrac{2W_1 + 9W_2}{3W_1 + 6W_2}\left(\dfrac{R-r}{g}\right)}.$

21. The Energy Equation for Reciprocating Engines.—The energy equation developed in Art. 19 finds an important practical application in the design of reciprocating engines as discussed in Arts. 17 and 18. We begin with the formulation of the expression for the total kinetic energy T of one crank, connecting rod, and piston as shown in Fig. 99 (see page 136). We recall that in previous discussions, the connecting rod of mass m' was replaced by two particles at its ends: m_1 at the crankpin A and m_2' at B. The latter mass m_2' was then combined with the mass m'' of the piston, giving the total mass $m_2 = m_2' + m''$ at B. We also assigned to the mass particle m_1 at A the fictitious moment of inertia

$$I = m'[i_1^2 - c(l-c)],$$

where i_1 is the radius of gyration of the connecting rod with respect to its centroidal axis, distance c from the crankpin A. The moment of inertia of the crank with respect to its axis of rotation was denoted by mi_0^2. With these notations, the total kinetic energy of the system, during motion, will be

$$T = \tfrac{1}{2}\{(mi_0^2 + m_1r^2)\dot\phi^2 + m_2\dot x^2 + m'[i_1^2 - c(l-c)]\dot\psi^2\}, \qquad (a)$$

where $\dot\phi$ is the angular velocity of the crank, $\dot x$ the linear velocity of the piston, and $\dot\psi$ the angular velocity of the connecting rod.

We have also seen previously that both $\dot{x}$ and $\dot{\psi}$ can be expressed in terms of ϕ. Differentiating expression (96), page 138, once with respect to time and using expression (m), page 141, as it stands, we have

$$\left.\begin{aligned} \dot{x} &= -r\dot{\phi}(\sin\phi + \tfrac{1}{2}A_2\sin 2\phi - \tfrac{1}{4}A_4\sin 4\phi \\ &\qquad\qquad + \tfrac{1}{6}A_6\sin 6\phi - + \cdots), \\ \dot{\psi} &= \lambda\dot{\phi}(C_1\cos\phi - \tfrac{1}{3}C_3\cos 3\phi + \tfrac{1}{5}C_5\cos 5\phi - + \cdots). \end{aligned}\right\} \qquad (b)$$

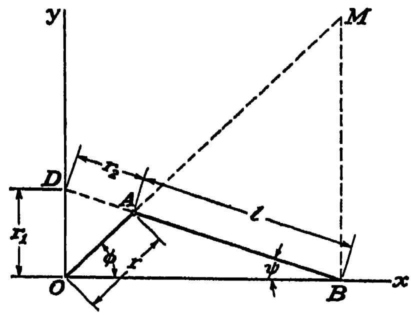


FIG. 138.

Substituting these expressions into Eq. (a) above, we obtain

$$T = \tfrac{1}{2}\Psi\dot{\phi}^2, \qquad (119)$$

in which Ψ is a rather cumbersome function of the angle ϕ and represents the so-called *reduced moment of inertia* of the system.

In practical applications, we usually define the function Ψ graphically. Consider, for example, a particular configuration of the system as shown in Fig. 138. Since M is the instantaneous center of rotation of the connecting rod, the absolute values of the velocities of points A and B are in the same ratio as the distances $\overline{AM}$ and $\overline{BM}$, *i.e.*,

$$\frac{|\dot{x}|}{r\dot{\phi}} = \frac{\overline{BM}}{\overline{AM}} = \frac{\overline{OD}}{\overline{OA}} = \frac{r_1}{r}.$$

and

$$|\dot{x}| = r_1\dot{\phi}. \qquad (c)$$

To find the angular velocity $\dot{\psi}$ of the connecting rod, we divide the velocity of point A by the distance $\overline{AM}$ and obtain $\dot{\psi} = r\dot{\phi}/\overline{AM}$, or since

$$\frac{\overline{AM}}{\overline{AB}} = \frac{\overline{AO}}{\overline{AD}} = \frac{r}{r_2},$$

we have

$$\dot{\psi} = \frac{r_2}{l}\,\dot{\phi}. \qquad (d)$$

If we repeat the construction in Fig. 138 for a number of values of ϕ and substitute each time in Eq. (a) the values of $\dot{x}$ and $\dot{\psi}$, computed from Eqs. (c) and (d), we can determine a series of values of the function Ψ for the chosen values of ϕ. In this way, the function Ψ can be represented graphically by a curve like the one shown in Fig. 139.[1] For an approxi-

[1] This curve was made for a Diesel engine having $\lambda = 0.41$.

mate calculation of Ψ, we can assume that the angle ψ is always small, neglect $\dot{\psi}$ entirely, and take as a first approximation for $\dot{x}$ [see Eqs. (*b*)]

$$\dot{x} \approx -r\dot{\phi} \sin \phi.$$

Then Eq. (*a*) for kinetic energy reduces to

$$T \approx \tfrac{1}{2}(mi_0^2 + m_1r^2 + m_2r^2\sin^2\phi)\dot{\phi}^2, \tag{e}$$

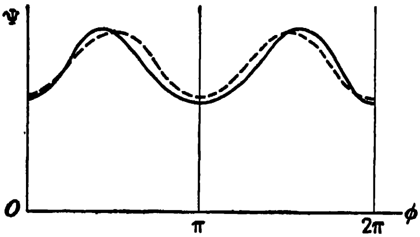


FIG. 139.

and the corresponding approximation for Ψ is represented by the function within the parentheses of this expression. It is shown graphically by the dotted curve in Fig. 139. We see that the function Ψ varies periodically with the angle of rotation ϕ of the crank, and in using expression (119), we are dealing with a system having a *variable moment of inertia* with respect to the axis of rotation through O.

Using expression (119) for kinetic energy, Eq. (108) becomes

$$d(\tfrac{1}{2}\Psi\dot{\phi}^2) = \Sigma F_e\cos(F_e,v)ds + \Sigma F_i\cos(F_i,v)ds. \tag{f}$$

The right-hand side of this equation represents the work of all forces acting on the various parts of the system during a small change in its configuration. In calculating this work, we have to consider, as external forces, the gravity forces and the resisting torque, equal and opposite to the shaft torque M_s transmitted through the shaft and capable of producing useful work,[1] and, as internal forces, the gas pressure in the cylinder and the friction between moving parts. Starting with the work of gas pressure in the cylinder, we dênote by P the resultant force exerted on the head of the piston. This resultant can be calculated for each position of the crank by using an indicator card diagram. For a small change in the configuration of the system as defined by $d\phi = \dot{\phi}\, dt$, the corresponding displacement of the piston is $dx = \dot{x}\, dt$. Using expression (*c*) above for $\dot{x}$, this becomes $dx = r_1d\phi$, and the work of the gas pressure is

$$Pr_1d\phi. \tag{g}$$

The moment Pr_1 represents the torque due to gas pressure and will be denoted by M_p. For each position of the crank, the length r_1 is found graphically as shown in Fig. 138, and the gas torque M_p can be represented by a curve as shown in Fig. 140 for a two-cycle engine. We see

[1] As we shall see later, this may include work done on a flywheel, which we do not consider as a part of the system.

that this torque is a periodic function of the angle rotation ϕ, repeating itself after each revolution of the crank.[1] The resisting torque

$$M_r = -M_s$$

acting on the shaft produces, during rotation $d\phi$, the work

$$-M_s d\phi. \tag{h}$$

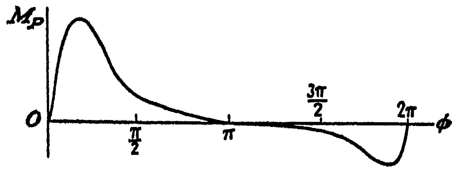


FIG. 140.

The work of friction forces during rotation $d\phi$ can be approximately calculated for each position of the crank if we know the pressures between the moving parts and the coefficients of friction.[2] This work can be represented in the form

$$-M_f d\phi, \tag{i}$$

where the friction torque M_f can also be represented graphically as a function of ϕ. Finally, the work of the gravity forces is obtained from the expression for the potential energy of the system and can be represented in the form

$$-W\, dy_c, \tag{j}$$

where W is the total weight of the system and y_c is the elevation of its mass-center. Substituting expressions (g), (h), (i), (j) all into Eq. (f), we obtain

$$d(\tfrac{1}{2}\Psi\dot{\phi}^2) = (M_p - M_s - M_f)d\phi - W\, dy_c. \tag{120}$$

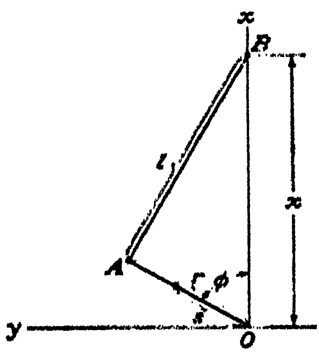


FIG. 141.

In the case of a horizontal engine as in Fig. 99, we have to consider only the gravity force mg of the crank concentrated at its center of gravity C and the gravity force m_1g of the portion of the connecting rod assigned to the crankpin A. The potential energy of these forces is

$$V = g(ms + m_1r)\sin\phi,$$

and their work, during rotation $d\phi$ of the crank, is

$$-\frac{dV}{d\phi}d\phi = -g(ms + m_1r)\cos\phi\, d\phi. \tag{k}$$

In the case of a vertical engine (Fig. 141), we have to consider not only the gravity forces mg and m_1g but also the gravity force m_2g for the

[1] In the case of a four-cycle engine, the gas torque repeats at intervals of 4π; see Fig. 148, p. 188.

[2] For a detailed discussion of these forces, see Biezeno and Grammel, "Technische Dynamik," p. 917, Springer, 1939.

mass particle at B. Measuring vertical displacements from the configuration corresponding to $\phi = 0$, the potential energy of the gravity forces, in this case, is

$$V = -g[(ms + m_1r)(1 - \cos\phi) + m_2(l + r - x)],$$

and their work, during rotation $d\phi$ of the crank, is

$$-\frac{dV}{d\phi}d\phi = g\left[(ms + m_1r)\sin\phi\, d\phi - m_2\frac{dx}{d\phi}d\phi\right].$$

Using expression (96), page 138, for x, this becomes

$$-\frac{dV}{d\phi}d\phi = g[(ms + m_1r + m_2r)\sin\phi$$
$$+ m_2r(\tfrac{1}{2}A_2\sin 2\phi - \tfrac{1}{4}A_4\sin 4\phi + \tfrac{1}{6}A_6\sin 6\phi - + \cdots)]d\phi \quad (l)$$

Comparing expressions (k) and (l) with Eqs. (98), page 139, we conclude that for a horizontal engine,

$$\frac{dV}{d\phi} = -\frac{g}{\omega^2}\int_{\frac{\pi}{2}}^{\phi} Y\, d\phi \qquad (m)$$

while, for a vertical engine,

$$\frac{dV}{d\phi} = -\frac{g}{\omega^2}\int_0^{\phi} X\, d\phi, \qquad (n)$$

where X and Y are the axial and transverse components, respectively, of the force transmitted to the foundation due to unbalance of the engine at uniform speed. Thus, if $Y = 0$ for the horizontal engine or if $X = 0$ for the vertical engine, the term $-W\, dy_c$ in Eq. (120) vanishes.

Integrating Eq. (120) for any initial angle of rotation ϕ_1 to some other value ϕ_2 and giving the corresponding values of the function Ψ the same subscript, we obtain the energy equation of the engine in the following form:

$$\tfrac{1}{2}(\Psi_2\dot{\phi}_2^2) - \tfrac{1}{2}(\Psi_1\dot{\phi}_1^2) = \int_{\phi_1}^{\phi_2}(M_p - M_s - M_f)d\phi + W(y_c' - y_c''), \quad (121)$$

where y_c' and y_c'' are the elevations of the mass-center corresponding to ϕ_1 and ϕ_2, respectively. If M_p, M_s, M_f, and Ψ are all known for each value of ϕ, the right-hand side of this equation can be evaluated, and we find the change in angular velocity $\dot{\phi}$ during rotation from ϕ_1 to ϕ_2. If we consider a complete revolution of the crank ($\phi_2 = \phi_1 + 2\pi$), the final and initial configurations coincide, the work of gravity forces vanishes, and $\Psi_2 = \Psi_1$. In such case, Eq. (121) becomes

$$\tfrac{1}{2}\Psi_1(\dot{\phi}_2^2 - \dot{\phi}_1^2) = \int_{\phi_1}^{\phi_1+2\pi}(M_p - M_s - M_f)d\phi. \qquad (122)$$

If the work done by the gas pressure during one revolution is just offset by the useful work together with that lost in friction, the integral on the right-hand side of Eq. (122) vanishes and we have $\dot{\phi}_2 = \dot{\phi}_1$. That is, the angular velocity of the engine, though not necessarily uniform, has the same value for angles of rotation ϕ that are multiples of 2π. This condition is referred to as the *steady state of motion* of the engine. If the work of the gas pressure per revolution exceeds the useful work together with that lost in friction, the angular velocity of the engine increases; under reversed conditions, the angular velocity decreases.

Having the general energy equation (121) for a single-cylinder engine, we can write a similar equation for an engine having several identical cylinders. For each cylinder, the values of Ψ, M_p, M_f, and

$$W(y_c' - y_c'')$$

are obtained from diagrams made for the single-cylinder engine. It is only necessary to take for each cylinder the proper value of ϕ. Then the energy equation for a multicylinder engine is obtained by summation as follows:

$$\tfrac{1}{2}\sum \Psi_2\dot{\phi}_2^2 - \tfrac{1}{2}\sum \Psi_1\dot{\phi}_1^2 = \int_{\phi_1}^{\phi_2}\left[\sum (M_p - M_f) - M_s\right]d\phi + \sum W(y_c' - y_c''). \quad (123)$$

Returning now to Eq. (120) and introducing the notation

$$M_g = W\frac{dy_c}{d\phi}, \quad (o)$$

which we call the *gravity torque*, we may write

$$d(\tfrac{1}{2}\Psi\dot{\phi}^2) = M\,d\phi, \quad (124)$$

where

$$M = M_p - M_s - M_f - M_g \quad (p)$$

is a torque that, during a change $d\phi$ in the coordinate ϕ, produces the same work as the actual forces acting on the system. We see that the engine is a system with one degree of freedom; its configuration is completely defined by the angle of rotation ϕ of the crank. This angle represents the *generalized coordinate* of the system, and the quantities Ψ and M are expressible as functions of this coordinate. The torque M, which produces the same work as the actual forces during a small change $d\phi$ in the generalized coordinate ϕ, is called the corresponding *generalized torque*. Equation (124) has the same form as an equation of motion for a body of variable moment of inertia Ψ rotating about a fixed axis under the action of a torque M and represents the most condensed form in which

the equation of motion of the engine can be expressed. Integrating this equation from some initial configuration ϕ_0 to any other configuration ϕ_1, we obtain

$$\tfrac{1}{2}\Psi_1\dot{\phi}_1^2 - \tfrac{1}{2}\Psi_0\dot{\phi}_0^2 = \int_{\phi_0}^{\phi_1} M\,d\phi. \tag{125}$$

From this equation, the angular velocity $\dot{\phi}$ of the crank can be found for any value of ϕ provided both Ψ and M are known as functions of ϕ and the initial angular velocity $\dot{\phi}_0$ is given.

If we divide both sides of Eq. (124) by dt, we have

$$\frac{d}{dt}\left(\frac{1}{2}\Psi\dot{\phi}^2\right) = M\dot{\phi}, \tag{q}$$

which states that the rate of change of kinetic energy of the system is equal to the rate at which the generalized force M produces work. Performing the indicated differentiation on the left-hand side of Eq. (q) and remembering that Ψ is a function of ϕ, we obtain

$$\frac{1}{2}\frac{d\Psi}{d\phi}\dot{\phi}^3 + \Psi\dot{\phi}\ddot{\phi} = M\dot{\phi}$$

or, after canceling $\dot{\phi}$,

$$\frac{1}{2}\frac{d\Psi}{d\phi}\dot{\phi}^2 + \Psi\ddot{\phi} = M. \tag{r}$$

The left-hand side of this equation can be represented in another form by using the expression

$$T = \tfrac{1}{2}\Psi\dot{\phi}^2 \tag{s}$$

for the kinetic energy of the system. Making the partial differentiations

$$\frac{\partial T}{\partial \phi} = \frac{1}{2}\frac{\partial \Psi}{\partial \phi}\dot{\phi}^2 \qquad \text{and} \qquad \frac{\partial T}{\partial \dot{\phi}} = \Psi\dot{\phi}, \tag{t}$$

together with the total differentiation

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\phi}}\right) = \frac{d}{dt}(\Psi\dot{\phi}) = \frac{d\Psi}{d\phi}\dot{\phi}^2 + \Psi\ddot{\phi}, \tag{u}$$

we see that Eq. (r) can be written in the form

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\phi}}\right) - \frac{\partial T}{\partial \phi} = M. \tag{126}$$

The left-hand side of this equation is obtained by differentiation of the general expression for the kinetic energy of the system. On the right-hand side, we have the generalized torque M corresponding to the selected generalized coordinate ϕ. If the kinetic energy T of the system

and the generalized torque M are known functions of ϕ, Eq. (126) can be written without difficulty. In this form, the equation of motion for the system is known as *Lagrange's equation.*[1]

We can readily generalize Eq. (126) to apply to any system that has a single degree of freedom. If the chosen generalized coordinate defining the configuration of the system is a linear coordinate s, then, instead of a corresponding generalized torque M, we shall have a generalized force F, and Eq. (126) takes the form

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{s}}\right) - \frac{\partial T}{\partial s} = F. \tag{126a}$$

In any case, the generalized force is always so chosen that during a small change ds in the generalized coordinate, it produces the same work as the actual forces acting on the system.

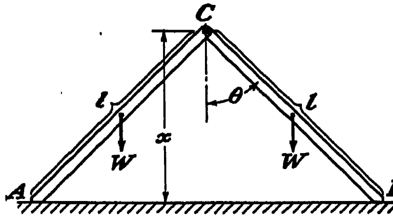


FIG. 142.

PROBLEMS

96. Two identical prismatic bars of weight W and length l are hinged together at C and supported by a smooth horizontal plane as shown in Fig. 142. Taking the angle θ as generalized coordinate, formulate the expressions for the corresponding generalized torque M and for the kinetic energy T of the system. Using these expressions in Eq. (126), show that the equation of motion of the system becomes

$$\ddot{\theta} = \frac{3g}{2l}\sin\theta.$$

97. Using Eq. (126a), write the equation of motion for the system in Fig. 142 when the vertical height x of point C is taken as the generalized coordinate.

$$\textit{Ans.}\quad \ddot{x} + \frac{x\dot{x}^2}{l^2 - x^2} + \frac{3g}{2l^2}(l^2 - x^2) = 0.$$

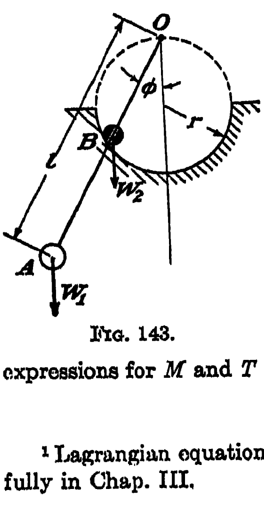


FIG. 143.

98. A simple pendulum of length l and weight W_1 is supported at point O and swings in a vertical plane as shown in Fig. 143. A second weight W_2, guided by a circular seat of radius r, can slide freely along the rod OA of the pendulum. Taking the angle ϕ as generalized coordinate, formulate the expressions for the corresponding generalized torque M and for the kinetic energy T of the system. Treat the weights W_1 and W_2 as dimensionless particles, and assume that the rod OA is without mass but completely inflexible; neglect all friction. Using the expressions for M and T in Eq. (126), write the equation of motion for the system.

$$\textit{Ans.}\quad \frac{W_1 l^2 + 4W_2 r^2}{g}\ddot{\phi} = -W_1 l\sin\phi - 2W_2 r\sin 2\phi.$$

[1] Lagrangian equations for systems with several degrees of freedom are discussed fully in Chap. III.

99. Write the equation of motion for the pendulum shown in Fig. 143 if the sliding weight W_2 follows a smooth horizontal surface distance h below point O. Make the same idealizing assumptions as in the preceding problem.

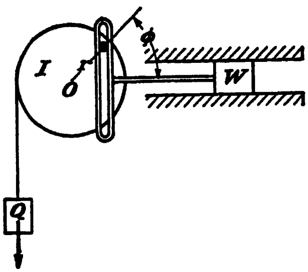


FIG. 144.

100. The system shown in Fig. 144 consists of a flywheel of moment of inertia I, a piston of weight W working from a crank of radius r, and a weight Q suspended from a string that is wound around the flywheel. Using ϕ as generalized coordinate, write the equation of motion for the system.

Ans. $\left(I + \frac{Q}{g}r^2 + \frac{W}{g}r^2\sin^2\phi\right)\ddot{\phi} + \frac{1}{2}\frac{W}{g}r^2\dot{\phi}^2\sin 2\phi = Qr.$

22. Flywheel Calculations.—As noted in the preceding article, the angular velocity of the crankshaft of an engine is likely to suffer considerable nonuniformity even in the steady state as a result of the variability of torque due to gas pressure. To minimize this nonuniformity of rotation and realize conditions satisfactory for the performance of useful work, it is common practice to use a flywheel that acts as an accumulator of energy. When the torque due to gas pressure exceeds the resisting torque, the speed increases and some of the work done is stored in the form of kinetic energy in the flywheel. This energy is then used in that portion of the cycle in which the resisting torque exceeds that due to gas pressure. The fluctuations in angular velocity will depend evidently on the size of the flywheel; and by increasing its moment of inertia, we can approach more and more closely to an absolutely uniform angular velocity. Usually, there are rather definite requirements regarding the uniformity of rotation for various kinds of machines. If $\dot{\phi}_{max}$ and $\dot{\phi}_{min}$ are the extreme values of the angular velocity of the shaft of an engine running in the steady state and $\dot{\phi}_a = \frac{1}{2}(\dot{\phi}_{max} + \dot{\phi}_{min})$ is its *average velocity*,[1] we define the grade of nonuniformity of rotation by the *coefficient of nonuniformity*

$$\epsilon = \frac{\dot{\phi}_{max} - \dot{\phi}_{min}}{\dot{\phi}_a}. \tag{a}$$

For machine shops, the usual requirements are $\epsilon = \frac{1}{35}$ to $\epsilon = \frac{1}{40}$, for direct-current generators, $\epsilon = \frac{1}{150}$ and, for alternating-current generators, $\epsilon = \frac{1}{300}$. For a given engine and a given requirement regarding ϵ, the general problem before us is to select the proper flywheel.

We begin with Eq. (120) of the preceding article, which we write in the more convenient form

$$d(\tfrac{1}{2}\Psi\dot{\phi}^2) = (M_p - M_s - M_f - M_g)d\phi, \tag{b}$$

[1] We assume that this is also the mean angular velocity.

where M_g is the torque producing, during rotation $d\phi$, the same work as the gravity forces. In the case of a horizontal engine, as we see from Eq. (k) of the preceding article,

$$M_g = g(ms + m_1r)\cos\phi.$$

For a vertical engine, the gravity torque M_g is obtained in a similar manner from Eq. (l) of the preceding article.

Performing the indicated differentiation on the left-hand side of Eq. (b) and remembering that Ψ is a function of ϕ, we have

$$d\left(\frac{1}{2}\Psi\dot{\phi}^2\right) = \frac{1}{2}\dot{\phi}^2\frac{d\Psi}{d\phi}d\phi + \Psi\dot{\phi}\frac{d\dot{\phi}}{d\phi}d\phi. \tag{c}$$

The first term of this differentiation represents the increase in kinetic energy of the engine due to change in the reduced moment of inertia Ψ. The second term is due to fluctuation in the angular velocity and is usually very small in comparison with the first term, so that it can be neglected without serious error, especially for high-speed engines. With this simplification and using $\dot{\phi}_a$ in place of $\dot{\phi}$, we write Eq. (b) in the form

$$\frac{1}{2}\dot{\phi}_a^2\frac{d\Psi}{d\phi}d\phi = (M_p - M_s - M_f - M_g)d\phi. \tag{d}$$

From this equation, we find that the shaft torque

$$M_s = M_p - M_f - M_g - \frac{1}{2}\dot{\phi}_a^2\frac{d\Psi}{d\phi}. \tag{e}$$

The calculations of the first three terms on the right-hand side of this expression and their representation as functions of ϕ have already been

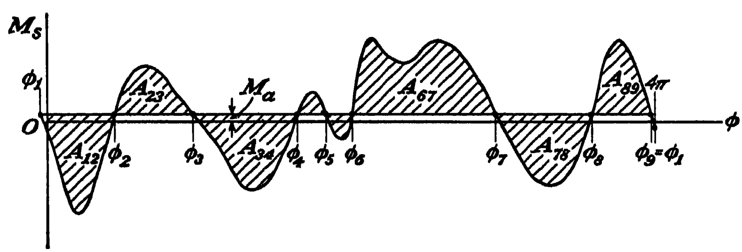


FIG. 145.

discussed in the preceding article. To evaluate the last term, we have to differentiate the Ψ-curve shown in Fig. 139. When all terms on the right-hand side of Eq. (e) have been evaluated, a curve for the shaft torque M_s can be drawn. Such a curve of M_s vs. ϕ for a four-cycle engine is shown in Fig. 145.[1] It will now be shown how this curve can

[1] The curves for this article have been taken from the book by Biezeno and Grammel, "Technische Dynamik," Springer, 1939.

be used for calculating the proper dimensions of a flywheel to ensure a desired grade of uniformity.

The torque M_s represents the turning moment transmitted through the crankshaft and capable of producing useful work. The area between this curve and the ϕ-axis represents the amount of useful work produced during every two revolutions of the engine. Dividing this work by 4π, we obtain the average value of M_s, which we denote by M_a. This average value of the shaft torque is represented in Fig. 145 by the horizontal line. Assume now that a flywheel is put on the shaft. Then the torque M_s transmitted through the shaft not only produces useful work but also accelerates and decelerates the flywheel. We assume that the useful work consumes a uniform torque. Then in the steady state, this torque is evidently equal to M_a. When M_s exceeds its average value M_a, the flywheel is accelerated; when M_s is less than M_a, the flywheel decelerates. Denoting the moment of inertia of the flywheel by I_w, we may write

$$I_w\ddot{\phi} = M_s - M_a, \tag{f}$$

where $\ddot{\phi}$ is the angular acceleration of the flywheel. Multiplying both sides of Eq. (f) by $\dot{\phi}\,dt = d\phi$, we obtain

$$\tfrac{1}{2}I_w d(\dot{\phi}^2) = (M_s - M_a)d\phi. \tag{g}$$

At the points of intersection of the M_s-curve with the M_a-line in Fig. 145, the acceleration $\ddot{\phi}$ vanishes, as we see from Eq. (f), and the angular velocity of the flywheel acquires its maximum or minimum value depending on the sign of $M_s - M_a$ in the preceding portion of the diagram. The angular positions of the crank at which these maximums and minimums occur are denoted by $\phi_2, \phi_3, \ldots$ as shown. Considering, for example, the configuration ϕ_2, we see that in the preceding portion of the diagram shown by the shaded area A_{12}, the quantity $M_s - M_a$ is negative; hence the flywheel decelerates, and ϕ_2 is a position of minimum angular velocity. By the same reasoning, we find that ϕ_3 is a position of maximum angular velocity, etc. The successive extreme values of $\dot{\phi}$ corresponding to $\phi_1, \phi_2, \phi_3, \ldots$ will be obtained by integrating Eq. (g). Applying this equation to an interval from ϕ_i to ϕ_{i+1} and integrating, we obtain

$$\tfrac{1}{2}I_w(\dot{\phi}_{i+1}^2 - \dot{\phi}_i^2) = \int_{\phi_i}^{\phi_{i+1}} (M_s - M_a)d\phi = A_{i,i+1}.$$

Since, in practice, the fluctuations in angular velocity are usually small, we can assume $(\dot{\phi}_{i+1} + \dot{\phi}_i)/2 = \dot{\phi}_a$ and write this equation in the simpler form

$$I_w\dot{\phi}_a(\dot{\phi}_{i+1} - \dot{\phi}_i) = A_{i,i+1}. \tag{h}$$

Now by using a planimeter, we determine the areas A_{12}, $A_{13} = A_{12} + A_{23}$,

$A_{14} = A_{13} + A_{34}, \ldots$ The last area A_{19} must vanish, since, by definition,

$$\int_{\phi_1}^{\phi_9} M_s d\phi = 4\pi M_a.$$

Having the areas $A_{12}, A_{13}, A_{14}, \ldots$, we select from among them $A_{\max}$ and $A_{\min}$. Then, on the basis of Eq. (h), we write

$$I_w \dot{\phi}_a(\dot{\phi}_{\max} - \dot{\phi}_{\min}) = A_{\max} - A_{\min}$$

or, using expression (a),

$$I_w \dot{\phi}_a{}^2 \epsilon = A_{\max} - A_{\min}.$$

If the required coefficient of nonuniformity is specified, we find, for the necessary moment of inertia of the flywheel,

$$I_w = \frac{A_{\max} - A_{\min}}{\dot{\phi}_a{}^2 \epsilon}. \tag{127}$$

It should be noted that Eq. (127) gives a somewhat exaggerated value for I_w, since, in its derivation, we neglected the second term in expression (c). This term

$$\Psi \dot{\phi} \frac{d\dot{\phi}}{d\phi} d\phi = \frac{1}{2} \Psi \frac{d(\dot{\phi}^2)}{d\phi} d\phi,$$

depending on the angular acceleration of the engine, represents a certain inertia action similar to that of a flywheel. If we take it into consideration, Eq. (g) above must be rewritten as follows:

$$\tfrac{1}{2} I_w d(\dot{\phi}^2) + \tfrac{1}{2} \Psi d(\dot{\phi}^2) = (M_s - M_a) d\phi. \tag{i}$$

Substituting for Ψ its average value

$$\Psi_a = \frac{1}{2\pi} \int_0^{2\pi} \Psi \, d\phi$$

and proceeding as before, we obtain, instead of formula (127), the more accurate formula for the moment of inertia of the flywheel

$$I_w = \frac{A_{\max} - A_{\min}}{\dot{\phi}_a{}^2 \epsilon} - \Psi_a. \tag{128}$$

Using either Eq. (127) or (128), we can find the required size of flywheel to obtain any desired grade of uniformity of rotation of the engine. We see, however, that the engine can never run with perfect uniformity unless the flywheel is infinitely large. Increasing the size of the flywheel, we can have a smaller and smaller grade of nonuniformity, but we can never eliminate it completely.

Theoretically, we can remove all nonuniformity by using a flywheel of

variable moment of inertia. To show this, we return to Eq. (b), which we write in the simplified form

$$d(\tfrac{1}{2}\Psi\dot{\phi}^2) = M\,d\phi, \tag{j}$$

where

$$M = M_p - M_s - M_f - M_g.$$

If, by some device, the angular velocity $\dot{\phi}$ is kept constant and equal to ω, then, from Eq. (j), we obtain

$$\frac{d\Psi}{d\phi} = \frac{2M}{\omega^2}. \tag{k}$$

We see that for absolute uniformity of rotation, the rate of change of the reduced moment of inertia Ψ must take place in proportion to the variable torque M. One method of accomplishing this is shown in Fig. 146 in which the radial distances s of sliding masses m on two spokes of the flywheel are made to vary as a proper function of the angle of rotation ϕ. In this way, the variation in the reduced moment of inertia Ψ can be made to satisfy Eq. (k). Naturally such a device works only as long as the torque M remains the same function of ϕ. Any change in this torque will require a corresponding change in the kinematical arrangement governing the radial distances s of the sliding masses; and for this reason, the device is not of much practical value.

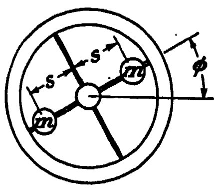


FIG. 146.

The foregoing method of flywheel calculation, using Eq. (127) or (128), can readily be extended to the case of an engine with several identical cylinders. In such case, to make a diagram like that shown in Fig. 145, we have to combine the actions of individual cylinders by taking proper account of the angular positions of the corresponding cranks. Assume, for example, that Fig. 147a represents the M_s-curve for one cylinder of a two-cycle engine. To get the M_s-curve for a two-cylinder engine, we observe that the two cranks are under the angle π. Dividing the curve in Fig. 147a into two parts and placing them as shown in Fig. 147b, we obtain, by superposition, the M_s-curve for a two-cylinder engine as shown by the heavy line. The complete period of the curve in this case is evidently π; *i.e.*, after every half revolution of the shaft, the ordinates of the M_s-curve repeat themselves.

Dividing the curve in Fig. 147a into three equal parts and superimposing as shown in Fig. 147c, we obtain the M_s-curve for a three-cylinder two-cycle engine. The complete period in this case is $2\pi/3$. Finally, in Fig. 147d, we have the M_s-curve for an eight-cylinder engine. This curve is obtained by dividing the curve in Fig. 147a into eight equal

parts and superimposing ordinates. We see that as the number of cylinders increases, the variation in M_s becomes less pronounced and the engine requires a smaller and smaller flywheel to attain a desired grade of uniformity of rotation.

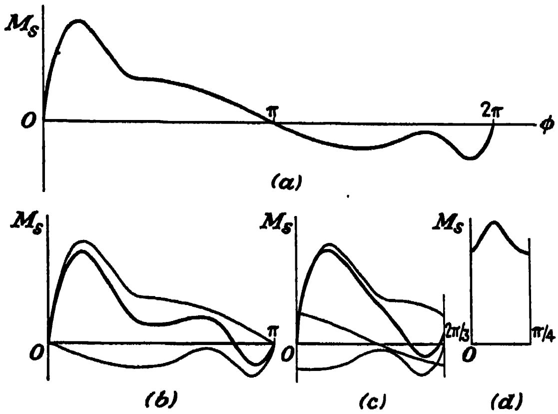


FIG. 147.

We have already observed that formula (127) for calculating the proper moment of inertia of the flywheel of an engine is not exact because, in its derivation, we neglected the second term in expression (*c*), which depends on the variation in angular velocity of the engine. This term was taken into account in developing formula (128), but this is still only an approximation, since the reduced moment of inertia Ψ was treated as a constant equal to its average value Ψ_a. Nevertheless, formulas (127) and (128) are sufficiently accurate for practical calculations if the variation in angular velocity is small ($\epsilon < \frac{1}{25}$) as it usually must be.

In those cases where ϵ is not small, a more accurate determination of the proper moment of inertia of a flywheel can be made by using Eq. (*b*) above. Solving this equation for M_s, we obtain

$$M_s = M_p - M_f - M_g - \frac{d}{d\phi}\left(\frac{1}{2}\Psi\dot{\phi}^2\right) \qquad (l)$$

instead of the approximate expression (*e*) previously obtained from Eq. (*d*). Using this more exact value of M_s in Eq. (*g*), we obtain

$$\frac{1}{2}I_w d(\dot{\phi}^2) = \left[M_p - M_f - M_g - \frac{1}{2}\frac{d}{d\phi}(\Psi\dot{\phi}^2) - M_a\right]d\phi. \qquad (m)$$

Dividing both sides of this equation by the square of the average angular velocity $\dot{\phi}_a$ and introducing the notation $z = \dot{\phi}^2/\dot{\phi}_a^2$, we obtain

$$\frac{d}{d\phi}[(I_w + \Psi)z] = \frac{2}{\dot{\phi}_a^2}(M_e - M_a), \qquad (n)$$

in which

$$M_e = M_p - M_f - M_g \tag{o}$$

represents the so-called *effective torque*. Integrating Eq. (n), we obtain

$$(I_w + \Psi)z - (I_w + \Psi_0)z_0 = \frac{2}{\phi_a^2}\int_0^\phi (M_e - M_a)d\phi, \tag{p}$$

where Ψ_0 and z_0 denote the values of Ψ and z corresponding to $\phi = 0$. For further use, we write Eq. (p) in the form

$$(I_w + \Psi)z = (I_w + \Psi_0)z_0 + \Phi, \tag{129}$$

where

$$\Phi = \frac{2}{\phi_a^2}\int_0^\phi (M_e - M_a)d\phi. \tag{q}$$

We consider now a graphical method of solution of Eq. (129) for the required moment of inertia I_w of a flywheel corresponding to a given permissible grade of non-uniformity of rotation. Assume, for example, that the fine-line curve in Fig. 148

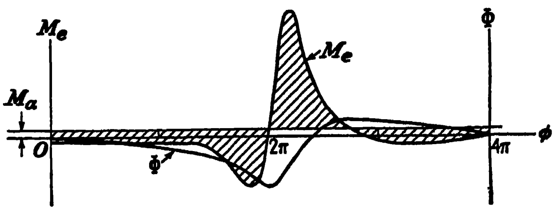


FIG. 148.

represents the effective turning moment M_e as a function of ϕ for a given four-cycle engine. Such a curve is readily obtained by superposition of the curves for M_p, M_f, and M_g, constructed as already explained in Art. 21. Since M_f and M_g are usually small, it will be found that the curve for M_e differs but little from that for M_p. Let M_a be the average value of M_e, and assume that this constant torque is that converted into useful work. Then, in such case, ordinates measured from the upper horizontal line in Fig. 148 represent the quantity $M_e - M_a$, and the corresponding integral curve, constructed as explained in Art. 2 and shown by the heavy line in Fig. 148, represents the function Φ as defined by Eq. (q).

Taking values of Φ from the curve in Fig. 148 for a series of uniformly spaced values of ϕ together with corresponding values of Ψ from the curve in Fig. 139, we now construct the closed curve shown in Fig. 149, which is called the *mass-force diagram*.[1] Any point on this curve represents simultaneous values of the functions Φ and Ψ appearing in Eq. (129), *i.e.*, values corresponding to the same value of ϕ. Let us suppose now, for the sake of further discussion, that we know both I_w and z_0 as well as Ψ_0 in Eq. (129), and let P be a point having the coordinates $-I_w$ and $-(I_w + \Psi_0)z_0$ in Fig. 149. Joining this point with any point Q on the closed $\Phi\Psi$-curve, we see, by reference to Eq. (129), that the slope of the line PQ is

$$\tan\theta = \frac{(I_w + \Psi_0)z_0 + \Phi}{I_w + \Psi} = z. \tag{r}$$

[1] See F. WITTENBAUER, "Graphische Dynamik," pp. 759, 775, Berlin, 1923.

From this expression, we may write

$$\sqrt{\tan \theta} = \sqrt{z} = \frac{\dot\phi}{\dot\phi_a}. \tag{s}$$

We see that the slope of each such line as PQ in Fig. 149 is directly related to the angular velocity $\dot\phi$ for that value of ϕ corresponding to the chosen point Q on the closed mass-force diagram. Drawing the enveloping tangents PA and PB as shown in the figure, we conclude accordingly that

$$\sqrt{\tan \alpha} = \frac{\dot\phi_{\max}}{\dot\phi_a} \quad \text{and} \quad \sqrt{\tan \beta} = \frac{\dot\phi_{\min}}{\dot\phi_a}. \tag{t}$$

Thus, we may write

$$\sqrt{\tan \alpha} - \sqrt{\tan \beta} = \frac{\dot\phi_{\max} - \dot\phi_{\min}}{\dot\phi_a} = \epsilon,$$

$$\frac{\sqrt{\tan \alpha} + \sqrt{\tan \beta}}{2} = \frac{\dot\phi_{\max} + \dot\phi_{\min}}{2\dot\phi_a} = 1,$$

from which

$$\left.\begin{aligned} \tan \alpha &= \left(1 + \frac{\epsilon}{2}\right)^2 \approx 1 + \epsilon, \\ \tan \beta &= \left(1 - \frac{\epsilon}{2}\right)^2 \approx 1 - \epsilon. \end{aligned}\right\} \tag{u}$$

We see now that knowing the specified grade of nonuniformity ϵ, we can draw the tangents AP and BP with the slopes defined by Eqs. (u), and their intersection P determines, by its abscissa, the required moment of inertia I_w of the flywheel.

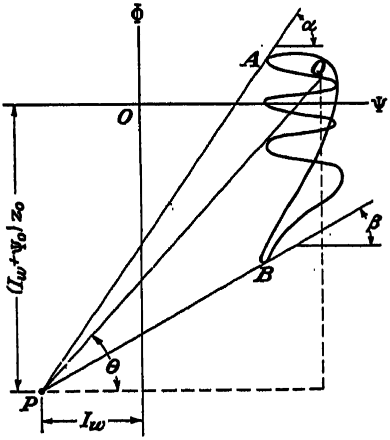


FIG. 149.

If the specified grade of nonuniformity ϵ is small, the lines AP and BP in Fig. 149 will be very nearly parallel and their intersection is not well defined. In such cases, it is preferable to use formula (127) or (128) to determine the required size of flywheel. The method illustrated in Fig. 149 should be reserved for use with relatively large values of ϵ.

CHAPTER III

DYNAMICS OF SYSTEMS WITH CONSTRAINTS

23. Equations of Constraint.—In Chap. II, dealing with the dynamics of a system of particles, we distinguished between *systems without constraints,* such as our solar system, and *systems with constraints,* such as the reciprocating engine in Fig. 99. In the one case, the particles are completely free and the paths that they describe will depend on the forces that may act on and within the system. Any change in these forces will produce corresponding changes in the trajectories of the various particles.

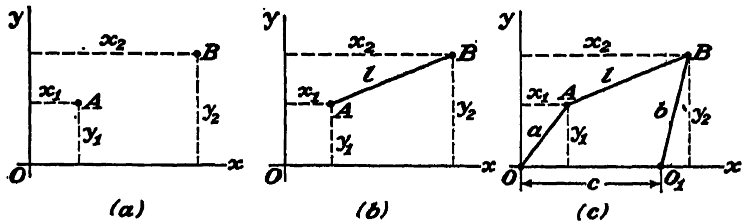


FIG. 150.

In the second case, the particles are not free but must follow prescribed paths depending on the nature of the constraints. Changes in the active forces of such a system may affect the velocities and accelerations of the particles but not their trajectories. Thus, the earth describes an ellipse around the sun only because it is the nature of the internal forces of gravitational attraction to produce such a motion. If some external force should be applied to the earth, its orbit would suffer a corresponding change. On the other hand, the piston of a reciprocating engine, guided by its cylinder wall, oscillates rectilinearly regardless of whether the gas pressure is a two- or a four-cycle affair. Systems with constraints are much more commonly encountered in engineering problems of dynamics than those without constraints. In this chapter, we shall develop some general dynamical principles especially suited to systems with constraints.

In general, if we have a system of n-particles without constraints, we find that $3n$ coordinates will be necessary to specify completely the configuration of the system. Ordinarily, these will be the $3n$ rectangular coordinates x_i, y_i, z_i for the n particles, and we say that the system has $3n$ *degrees of freedom.* Consider, for example, the case of two particles A and B as shown in Fig. 150a. If these particles are completely free in

space, we need the six coordinates x_1, y_1, z_1 and x_2, y_2, z_2 to define the configuration of the system; *i.e.*, these are six degrees of freedom. If the particles are constrained to remain in the xy-plane, we need only four coordinates x_1, y_1 and x_2, y_2, and the system has four degrees of freedom. Let us assume now that the particles, besides being confined to the xy-plane, are connected together by a rigid but weightless bar of length l (Fig. 150*b*). In such case, the four coordinates x_1, y_1 and x_2, y_2, are no longer independent but must always be such as to satisfy the geometric relationship

$$(x_2 - x_1)^2 + (y_2 - y_1)^2 = l^2. \tag{a}$$

Thus if three of the coordinates are specified, the fourth one can be computed from Eq. (*a*) and we have a system with three degrees of freedom. Finally, if besides the interconnecting bar AB we have also the rigid but weighless bars OA and O_1B connecting the particles to fixed points O and O_1 in the xy-plane, we must have, in addition to Eq. (*a*) above, the further geometric requirements

$$\left.\begin{aligned} x_1^2 + y_1^2 &= a^2, \\ (x_2 - c)^2 + y_2^2 &= b^2. \end{aligned}\right\} \tag{b}$$

In such case, if we specify any one of the four coordinates, the other three will be found from Eqs. (*a*) and (*b*) and the system has only one degree of freedom.

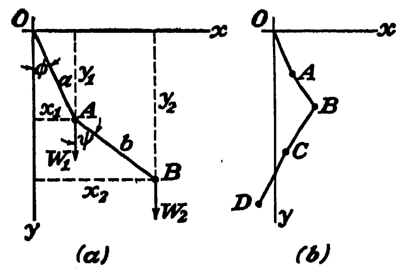


FIG. 151.

As another example of constraint, let us consider the double pendulum swinging in the vertical xy-plane as shown in Fig. 151*a*. Here again, we have two particles A and B connected together by a bar of length b and attached to a fixed point O by a bar of length a. To define the configuration of this system in the xy-plane, we take the four coordinates x_1, y_1 and x_2, y_2, which must satisfy the geometric requirements

$$\left.\begin{aligned} x_1^2 + y_1^2 &= a^2, \\ (x_2 - x_1)^2 + (y_2 - y_1)^2 &= b^2, \end{aligned}\right\} \tag{c}$$

and the system has two degrees of freedom. For the case shown in Fig. 151*b*, we have four particles suspended from a fixed point O and connected by four rigid bars. Altogether, eight coordinates are required to specify the configuration of this system; but since we can write four equations similar to Eqs. (*c*), we conclude that the system has four degrees of freedom.

Equations such as (*a*), (*b*), and (*c*), limiting the independence of the coordinates that specify the configuration of a system of particles, are called *equations of constraint.* Any kind of constraints introduced into a system will limit the freedom of motion of its particles and therefore reduce the number of degrees of freedom. In general, if there are n particles in the system and k equations of constraint, the number of degrees of freedom will be $3n - k$. That is, we need to specify only $3n - k$ of the necessary $3n$ coordinates to define the configuration of the system; the remaining k coordinates will be determined by the equations of constraint. If the system of n particles is confined to one plane as is so often the case, the number of degrees of freedom in that plane is $2n - k$. In such case, of course, we are simply taking for granted n equations of constraint of the type $z_i = 0$.

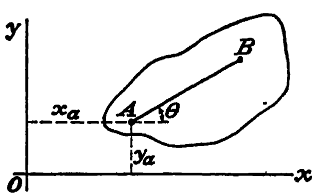


FIG. 152.

Practically, there are many kinds of physical constraints to which a system of particles may be subjected. Very often, the constraints are such that all possibility of relative motion between the particles is eliminated and we have the case of a *rigid body.* To define the position of a rigid body in space, we need the coordinates x_a, y_a, z_a of an arbitrary point A of the body and three angles defining rotation of the body with respect to that point.[1] Thus the free rigid body is a system of particles with six degrees of freedom. If the body is confined to move parallel to a fixed plane, it has three degrees of freedom. In such case, the coordinates necessary to define the position of the body in the xy-plane of motion are x_a, y_a, and θ as shown in Fig. 152.

Every machine can be considered as a system of rigid bodies interconnected between themselves and the foundation by such constraints as bearings, guides, cams, gears, chains, etc., and all these constraints limit the number of degrees of freedom that the system may have. We have already seen, for example, how the reciprocating engine in Fig. 99 (see page 136) is a system with but one degree of freedom.

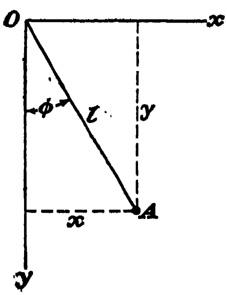


FIG. 153.

In the preceding examples, we see that the equations of constraint such as Eqs. (*a*), (*b*), and (*c*) involve only coordinates of the system and certain constants. Although such constraints are the most common, we can also have cases where the equations of constraint will involve time. Consider, for example, the simple pendulum shown in Fig. 153. If the

[1] The proper selection of these angles is discussed in Chap. V.

suspension point O is fixed and the particle A is constrained to move only in the plane of the figure, we have, as an equation of constraint,

$$x^2 + y^2 = l^2, \tag{d}$$

which involves only coordinates x and y and the constant l. Suppose now that the point O is not fixed but oscillates vertically with a known simple harmonic motion

$$y_0 = a \sin pt.$$

Then, instead of Eq. (d), we have as an equation of constraint

$$x^2 + (y - a \sin pt)^2 = l^2, \tag{e}$$

which involves time. Systems with constraints that involve time will be discussed again in more detail in Art. 29.

Sometimes the constraints of a system are such that the equations of constraint contain not only coordinates but also their derivatives with respect to time. As an example of such constraints, let us consider a thin circular disk of radius a rolling in its own vertical plane along a linear track OD, which we take as the x-axis (Fig. 154). In general, the position of the disk is defined by the coordinate x_c of its center C and the angle of rotation ϕ. If there is no friction between the disk and its track, the only equation of constraint is

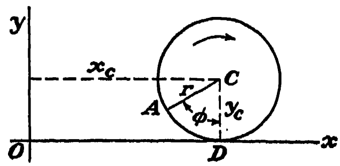


FIG. 154.

$$y_c = a \tag{f}$$

and we have a system with two degrees of freedom. Let us assume now that there is friction sufficient to prevent slipping at the point of contact D. Then the velocity of that point must vanish, and we have the additional equation of constraint

$$\dot{x}_c = a\dot{\phi}. \tag{g}$$

which contains derivatives of the coordinates with respect to time. In this particular case, we can readily integrate expression (g) and obtain

$$x_c = a\phi + c, \tag{h}$$

which again involves only the coordinates themselves and the constant c. Sometimes, equations of constraint containing velocities cannot be integrated as above, and then the problem is more complicated. Examples of this kind will be considered again in more detail in Art. 32 at the end of this chapter.

24. Generalized Coordinates and Generalized Forces.—In our previous discussions of systems of particles, we used, for the most part,

rectangular coordinates x, y, z. In the case of systems with constraints, these coordinates are not independent but are connected by equations of constraint, examples of which were discussed in the preceding article. In such cases, it is advantageous in defining the configuration of a system to use, instead of rectangular coordinates, some quantities that are independent of each other. Such quantities are called *generalized coordinates*.

Considering, for example, the theoretical pendulum (Fig. 153), it is advantageous to take, instead of x and y coordinates, the angle ϕ as the quantity defining the configuration of the system. This angle will be the generalized coordinate. Knowing ϕ, we can determine the rectangular coordinates of the particle A by using the equations

$$\left.\begin{aligned} x &= l \sin \phi, \\ y &= l \cos \phi. \end{aligned}\right\} \tag{a}$$

In discussing the motion of a reciprocating engine (Fig. 99), we found that the configuration of the system is completely defined by the angle of rotation ϕ of the crank. Hence this angle can be taken as the generalized coordinate.

In the case of a double pendulum (Fig. 151a), the configuration of the system is completely defined by the angles ϕ and ψ, which are independent of each other and can be taken as generalized coordinates. Knowing these quantities, we can calculate the rectangular coordinates of the two particles A and B by using the equations

$$\left.\begin{aligned} x_1 &= a \sin \phi, & y_1 &= a \cos \phi, \\ x_2 &= a \sin \phi + b \sin \psi, & y_2 &= a \cos \phi + b \cos \psi. \end{aligned}\right\} \tag{b}$$

As another example, we take the device shown in Fig. 155. This system of bars, hinged to a fixed point O, can move only in the xy-plane. It is seen that the configuration of the system, at any instant, is completely defined if we know the angles ϕ and ψ, which can be taken as generalized coordinates in this case.

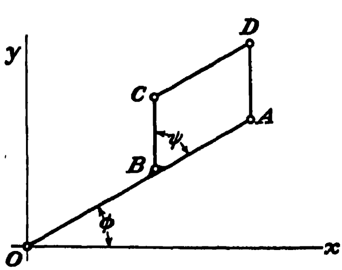


FIG. 155.

It is evident that in each case, the number of generalized coordinates will be equal to the number of the degrees of freedom of the system. For example, as we have already noted, the system in Fig. 151b has four degrees of freedom. To define the configuration of this system, we need four quantities; and as generalized coordinates, we can take the four angles that the bars of the system make with the y-axis.

In correspondence with *generalized coordinates*, we introduce now the notion of *generalized forces*.[1] Let us consider first a general case and assume that the configuration of a system with n degrees of freedom is defined by the generalized coordinates $q_1, q_2, \ldots q_n$. All these coordinates, by definition, are independent. We can give to one of them, q_i, a small increment δq_i without changing the magnitudes of others. To this change of the coordinate q_i there will correspond a certain displacement of the system, and the forces acting thereon will produce the work

$$\delta q_i Q_i,$$

where Q_i will be a certain expression containing forces acting on the system. This expression can be readily derived in each particular case and represents the *generalized force* corresponding to the generalized coordinate q_i. We can state that the generalized force, corresponding to the generalized coordinate q_i, is that quantity by which we must multiply the increment δq_i of the coordinate in order to obtain the work produced by the acting forces during the displacements corresponding to the assumed change in q_i.

Take, for example, the double pendulum, shown in Fig. 151*a*. If we give to the generalized coordinate ψ a small increment $\delta\psi$, the particle B will be raised by the amount $b \sin \psi \, \delta\psi$ and the gravity force W_2 will produce the work

$$-W_2 b \sin \psi \, \delta\psi.$$

Thus, in accordance with the above definition, the quantity

$$\Psi = -W_2 b \sin \psi \tag{c}$$

is the generalized force corresponding to the generalized coordinate ψ. To obtain the generalized force corresponding to the coordinate ϕ, we give to this coordinate a small increment $\delta\phi$ and, at the same time, keep the angle ψ unchanged. In such case, the vertical displacement of each particle is equal to $a \sin \phi \, \delta\phi$ and the work done by the gravity forces is

$$-(W_1 + W_2) a \sin \phi \, \delta\phi.$$

Hence the generalized force in this case is

$$\Phi = -(W_1 + W_2) a \sin \phi. \tag{d}$$

We see that in both cases (*c*) and (*d*), the generalized force has the dimension of force multiplied by length. This is as it should be, since the increments of the coordinates $\delta\phi$ and $\delta\psi$ are pure numbers and their products with the corresponding generalized forces must give work.

[1] The notions of generalized coordinates and generalized forces were introduced in mechanics by Lagrange, who uses them widely in his famous book "Mécanique analytique," Paris, 1788.

As another example, let us consider the pendulum shown in Fig. 156. This system, which has been used to record the oscillations of ships, consists of a weighted bar DC free to slide in a sleeve that can rotate freely about a fixed point O. An intermediate point B of this bar is connected to another bar AB of length r, which is hinged to the fixed point A at the distance h vertically below point O. Other dimensions are as shown in the figure. This is a system with one degree of freedom, and its configuration will be completely defined by the angle of rotation ϕ that the bar AB makes with the vertical. Taking this angle as our generalized coordinate, the y-coordinate of the point C is

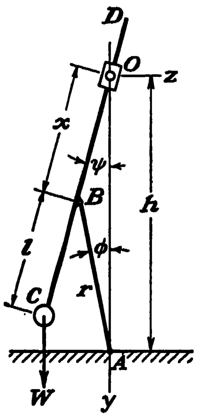


FIG. 156.

$$y_c = h - r\cos\phi + l\cos\psi.$$

We have also from geometrical considerations

$$r\sin(\phi + \psi) = h\sin\psi.$$

Assuming now that the angles ϕ and ψ are always small, we may write

$$r(\phi + \psi) \approx h\psi, \qquad \psi = \frac{r\phi}{h - r}.$$

Then

$$y_c \approx h - r\left(1 - \frac{\phi^2}{2}\right) + l\left(1 - \frac{\psi^2}{2}\right)$$

$$= h + l - r + \frac{r}{2}\left[1 - \frac{rl}{(h - r)^2}\right]\phi^2. \qquad (e)$$

We give now to ϕ a small increase $\delta\phi$. Then the weight W moves down by the amount

$$\frac{dy_c}{d\phi}\,\delta\phi = r\left[1 - \frac{rl}{(h - r)^2}\right]\phi\,\delta\phi$$

and produces the work

$$Wr\left[1 - \frac{rl}{(h - r)^2}\right]\phi\,\delta\phi.$$

Hence the required generalized force, corresponding to ϕ, is

$$\Phi = Wr\left[1 - \frac{rl}{(h - r)^2}\right]\phi. \qquad (f)$$

In each of the foregoing examples, it will be noted that in calculating the work of acting forces corresponding to an assumed virtual displacement of the system, we have considered only external forces. But there are also internal forces such as the axial forces in the bars of the pendu-

lums in Fig. 151 and sliding friction in the sleeve of the system in Fig. 156. In neglecting the work of such forces during an assumed virtual displacement of the system, we have tacitly assumed that the connecting bars are absolutely rigid so that their lengths do not change and also that there is no friction in sleeves, hinges, etc. Very often, physical connecting bars are sufficiently rigid and sleeves and hinges sufficiently free from friction to justify such procedure. Systems involving such *ideal constraints* (absolutely rigid and without friction) are called *ideal systems*. We shall usually assume, as above, that we have such systems; and in dealing with them, we shall find that the work of internal forces always vanishes for any virtual displacement of the system. Thus the expressions for generalized forces contain only external forces.

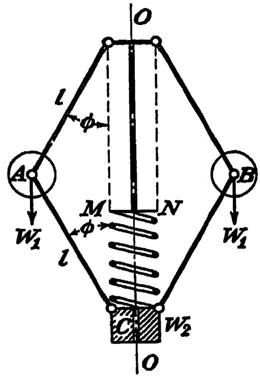


FIG. 157.

If, in addition to absolutely rigid constraints, a system contains also elastic bars or springs, the internal forces in such members may produce work during a virtual displacement. In such case, to obtain an ideal system, we simply assume the elastic bars or springs to be cut and replace them by the forces that they exert on the remainder of the system. Thereafter, we treat these forces exactly like external forces.

As an example of a system involving a spring, let us consider the governor shown in Fig. 157. This system, rotating about the vertical axis OO, consists of two flyballs connected to the rotating shaft and to a sliding weight by rigid bars. Any increase in the radial distances of the balls is resisted by an internal spring as shown. The weight of each flyball we denote by W_1 and the sliding weight by W_2. We neglect the weights of the spring and bars and assume that there is no torque acting on the vertical axis OO and no force in the spring when the connecting bars are vertical, *i.e.*, when $\phi = 0$. Then for the configuration shown in the figure, the spring is compressed by the amount $2l(1 - \cos\phi)$. Denoting the spring constant by k, we find that the compressive force in the spring is $2l(1 - \cos\phi)k$. The configuration of the system is completely defined by the angle ϕ and the angle of rotation θ of the governor with respect to its vertical axis OO. These two angles will be taken as the generalized coordinates of the system. To find the generalized force corresponding to the coordinate ϕ, we give to this coordinate a small increment $\delta\phi$. Due to this increment, the flyballs will be raised by the amount $l \sin\phi\, \delta\phi$ and the sliding weight by the amount $2l \sin\phi\, \delta\phi$, which is also the increment of compression in the spring. The work of all

forces acting on the system, corresponding to the assumed displacement, is

$$-[2W_1l \sin \phi + 2W_2l \sin \phi + 2l(1 - \cos \phi)k2l \sin \phi]\delta\phi.$$

Hence the required generalized force is

$$\Phi = -2l \sin \phi[W_1 + W_2 + 2l(1 - \cos \phi)k]. \qquad (g)$$

If we give a small increment $\delta\theta$ to the angle of rotation θ, the acting forces, which all are vertical, do not produce any work on the corresponding displacements. This indicates that the generalized force Θ, corresponding to the coordinate θ, is zero in this case.

The calculation of generalized forces in the above examples can be simplified if we observe that in all cases, the forces have potential. Taking first the general case, assume that the generalized coordinates, defining the configuration of the system, are $q_1, q_2, \ldots q_n$ and that the potential energy of the system is represented as a function of these coordinates, *i.e.*,

$$V = f(q_1, q_2, \ldots q_n).$$

If we give to one of the coordinates, q_i, a small increment δq_i, the corresponding increment in the potential energy is

$$\frac{\partial V}{\partial q_i}\delta q_i.$$

From the definition of potential energy (see Art. 20), we know that this increment is equal in magnitude and opposite in sign to the work done by the acting forces during the displacements corresponding to the increment δq_i. Denoting by Q_i the generalized force corresponding to q_i, we obtain

$$\frac{\partial V}{\partial q_i}\delta q_i = -Q_i\delta q_i.$$

From this equation, we have

$$Q_i = -\frac{\partial V}{\partial q_i}. \qquad (130)$$

Let us apply this expression for generalized force first to the example of the double pendulum (Fig. 151a). In this case, we have to consider only the gravity forces W_1 and W_2. Taking the potential energy equal to zero when the particles A and B are in their lowest positions, we find that for the configuration shown in the figure, the potential energy of the system is

$$V = W_1a(1 - \cos \phi) + W_2[a(1 - \cos \phi) + b(1 - \cos \psi)].$$

Then using Eq. (130), we find that the generalized forces corresponding to the coordinates ψ and ϕ are

$$\Psi = -\frac{\partial V}{\partial \psi} = -W_2 b \sin \psi,$$
$$\Phi = -\frac{\partial V}{\partial \phi} = -(W_1 + W_2)a \sin \phi,$$

which coincide with expressions (c) and (d) obtained before.

In the case of the system in Fig. 156, we use expression (e) for the coordinate y_c and find that the potential energy of the ball is

$$V = -W\left\{h + l - r + \frac{r}{2}\left[1 - \frac{rl}{(h-r)^2}\right]\phi^2\right\}.$$

Differentiating this expression in accordance with Eq. (130), we obtain expression (f) for the generalized force Φ.

Finally, in the case of the governor in Fig. 157, the potential energy of the gravity forces is

$$2W_1 l(1 - \cos \phi) + 2W_2 l(1 - \cos \phi),$$

and the potential energy of the spring is

$$\tfrac{1}{2}4l^2(1 - \cos \phi)^2 k.$$

Hence, the total potential energy of the system is

$$V = 2l(1 - \cos \phi)(W_1 + W_2) + 2l^2(1 - \cos \phi)^2 k.$$

Calculating the derivative of this expression with respect to ϕ, we find the corresponding generalized force given by Eq. (g).

PROBLEMS

101. The angle of rotation ϕ of the lever AB in Fig. 158 is taken as the generalized coordinate. Find the expression for the corresponding generalized force. *Ans.* $\Phi = -W_1 a + W_2 b$.

102. A vertical circular shaft AB, the torsional rigidity of which is GJ, carries two disks as shown in Fig. 159. Taking the angles of rotation ϕ and ψ of the disks as generalized coordinates,

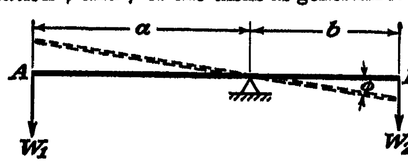


FIG. 158.

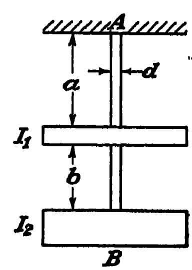


FIG. 159.

find the expressions for the corresponding generalized forces.

HINT: The potential energy of torsion is

$$V = \frac{\phi^2 GJ}{2a} + \frac{(\psi - \phi)^2 GJ}{2b}.$$

The generalized forces are obtained by using Eq. (130).

103. Two particles of weights W_1 and W_2 are vertically suspended on springs with spring constants k_1 and k_2 as shown in Fig. 160. Taking, as generalized coordinates, the vertical displacements y_1 and y_2 of the particles from their equilibrium positions, find the corresponding generalized forces.

Ans. $Y_1 = (y_2 - y_1)k_2 - y_1k_1;\ Y_2 = -(y_2 - y_1)k_2.$

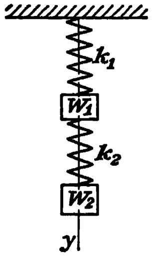


FIG. 160.

25. Equations of Equilibrium in Generalized Coordinates.—In discussing the conditions of equilibrium of ideal systems, we shall use the principle of virtual displacements.[1] Considering a general case, we assume that $q_1, q_2, \ldots q_n$ are generalized coordinates and $Q_1, Q_2, \ldots Q_n$, the corresponding generalized forces. Since the coordinates $q_1, q_2 \ldots q_n$ completely define the configuration of the system, any desired virtual displacement can be accomplished by giving to these coordinates the properly selected increments $\delta q_1, \delta q_2, \ldots \delta q_n$. During this displacement, the forces acting on the system produce the work

$$Q_1\delta q_1 + Q_2\delta q_2 + \cdots + Q_n\delta q_n.$$

If the system is in equilibrium, this work must vanish for every assumed virtual displacement and we obtain the equation

$$Q_1\delta q_1 + Q_2\delta q_2 + \cdots + Q_n\delta q_n = 0. \qquad (a)$$

This equation must be satisfied for any assumed values of $\delta q_1, \delta q_2, \ldots .$ We can assume that all increments of the coordinates except δq_i vanish. Then Eq. (a) gives

$$Q_i\delta q_i = 0;$$

and since δq_i is different from zero, we conclude that for equilibrium we must have

$$Q_i = 0.$$

Such an equation can be written for every generalized coordinate, and we obtain the following equations of equilibrium of the system

$$Q_1 = 0, \qquad Q_2 = 0, \qquad \cdots \qquad Q_n = 0. \qquad (131)$$

To have equilibrium all generalized forces must vanish. The number of Eqs. (131) is equal to the number of coordinates, and from them the values of the coordinates, defining the configurations of equilibrium, can be calculated. Take, for example, the double pendulum in Fig. 151a. The generalized forces for this case are given by Eqs. (c) and (d) of the preceding article. Thus, the equations of equilibrium are

$$W_2 b \sin\psi = 0,$$
$$(W_1 + W_2)a \sin\phi = 0.$$

[1] See the authors' "Engineering Mechanics," 2d ed., p. 217.

These equations require that

$$\sin\psi = \sin\phi = 0.$$

Hence ϕ and ψ must vanish or be equal to π. The corresponding configurations of equilibrium are shown in Fig. 161.

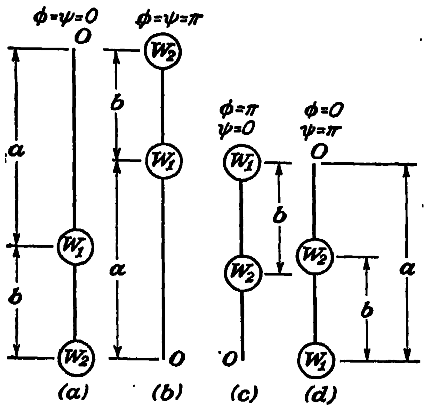


FIG. 161.

In the case of the governor shown in Fig. 157, the generalized force corresponding to the angle ϕ is given by Eq. (*g*) of the preceding article, and the equation of equilibrium is

$$2l\sin\phi[W_1 + W_2 + 2l(1-\cos\phi)k] = 0.$$

Since the expression within the brackets is always positive, we must have $\sin\phi = 0$; *i.e.*, $\phi = 0$, or $\phi = \pi$. From Fig. 157, we see that there is only one possibility, namely: $\phi = 0$.

If the forces have potential, the expressions for the generalized forces are obtained from Eq. (130) and the equations of equilibrium (131) take the form

$$\frac{\partial V}{\partial q_1} = 0, \qquad \frac{\partial V}{\partial q_2} = 0, \qquad \cdots \qquad \frac{\partial V}{\partial q_n} = 0. \tag{132}$$

We see that such equations of equilibrium are the same as those used in calculating the maximum or minimum of a function. This observation can be useful, in any given case, in deciding whether the equilibrium is *stable* or *unstable*. If the potential energy in the configuration of equilibrium has a minimum, this configuration is one of stable equilibrium.[1] This follows from the fact that for any adjacent configuration, the potential energy will be larger than for the position of equilibrium. Hence, some positive work will be required to bring the system to that configuration. If the potential energy of the system in the position of equilibrium is not a minimum, the equilibrium is, generally speaking, unstable.[2] Considering, for example, the configurations shown in Fig.

[1] This theorem was established by Lagrange in his "Mécanique analytique," vol. 1, pt. 1, sec. 3, Art. 5, 1788. A more rigorous proof of the same theorem was given by Lejeune-Dirichlet, *Jour. für Math.*, vol. 32, p. 85, 1846.

[2] A detailed study of this question was given by A. M. Liapounoff, *Jour. de math.*, ser. 5, vol. 3, p. 81, 1897.

161, we can conclude that only the configuration (a) is stable, since only in that case does the potential energy have a minimum.

As another example, let us consider the pendulum shown in Fig. 156. Measuring the vertical distance of the weight W from the horizontal plane through the fixed point A and using expression (e) of the preceding article, we find, for the potential energy of the system, the expression

$$V = W(h - y_c) = W\left\{r - l - \frac{r}{2}\left[1 - \frac{rl}{(h-r)^2}\right]\phi^2\right\}. \tag{b}$$

Then to obtain the configuration of equilibrium, we use Eqs. (132), which gives

$$\frac{dV}{d\phi} = -Wr\left[1 - \frac{rl}{(h-r)^2}\right]\phi = 0. \tag{c}$$

Assuming that the expression in the brackets is different from zero, we conclude that for equilibrium we must have $\phi = 0$. To decide if this configuration represents a maximum or a minimum of the potential energy (b), we calculate the second derivative

$$\frac{d^2V}{d\phi^2} = -Wr\left[1 - \frac{rl}{(h-r)^2}\right].$$

If this derivative is positive, the potential energy is a minimum and the equilibrium is stable. Hence for stability we must have

$$\frac{rl}{(h-r)^2} > 1$$

or

$$\sqrt{rl} > h - r.$$

If

$$\sqrt{rl} < h - r,$$

the potential energy for the position of equilibrium becomes a maximum and the weight W descends for any small increase of the angle ϕ. In such case, the position of equilibrium represented by $\phi = 0$ is unstable and any slight disturbance will cause the angle ϕ to increase. In the particular case when

$$\sqrt{rl} = h - r,$$

the equilibrium is indifferent; and as we see from expression (c), the system can remain at rest for any small value of ϕ.

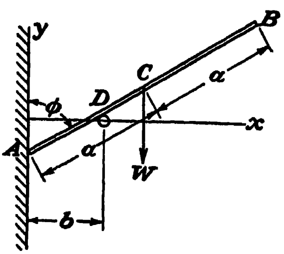


FIG. 162.

As another example, let us consider the conditions of equilibrium of the prismatic bar AB supported in a vertical plane as shown in Fig. 162.

We assume that both the peg D and the vertical wall at A are without friction. Taking the angle ϕ as the generalized coordinate and selecting the rectangular coordinates as shown, the coordinate y_c of the center of gravity C is

$$y_c = a \cos \phi - b \cot \phi,$$

and the potential energy of the system is

$$V = W(a \cos \phi - b \cot \phi).$$

The position of equilibrium is found from the equation

$$\frac{dV}{d\phi} = -W\left(a \sin \phi - \frac{b}{\sin^2\phi}\right) = 0,$$

which gives

$$\sin^3\phi = \frac{b}{a}.$$

Making the second derivative

$$\frac{d^2V}{d\phi^2} = -W\left(a \cos \phi + \frac{2b \cos \phi}{\sin^3\phi}\right),$$

which is negative, we conclude that the equilibrium is unstable, since the corresponding V is a maximum.

PROBLEMS

104. Six identical bars each of weight W form a hinged hexagon suspended at A as shown in Fig. 163. Find the forces P and Q that will keep the hexagon from altering its shape. *Ans.* $P = 5\sqrt{3}W/2$, $Q = \sqrt{3}W/2$.

105. Investigate the condition for which the metronome, shown in Fig. 164 will be in a condition of stable equilibrium. *Ans.* $W_1a - W_2b > 0$.

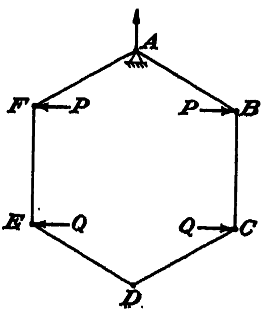


FIG. 163.

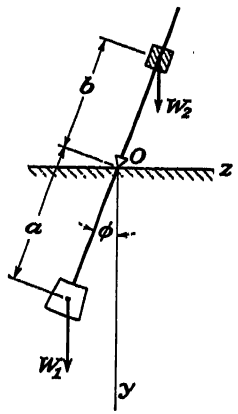


FIG. 164.

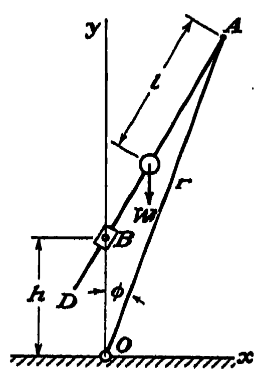


FIG. 165.

106. Investigate the condition of equilibrium of the pendulum shown in Fig. 165 and similar to that in Fig. 156. Points O and B are fixed points, and the bars OA and

AD are to be considered as weightless. Assume also that the angle ϕ defining the configuration of the system is always small.

26. Application of Generalized Coordinates in Bending of Beams.—Generalized coordinates can be used to advantage in investigating deflections of beams. We shall consider first the case of a beam with simply supported ends carrying a concentrated load P (Fig. 166). The deflection curve may have an infinite number of shapes; and to define the configuration of equilibrium of the system, we need an infinite number of coordinates. We always can represent the actual deflection curve by superposition of the simple sinusoidal curves shown in Fig. 166*b*, *c*, *d*, etc., and use, for the deflection at any cross section of the beam, the series

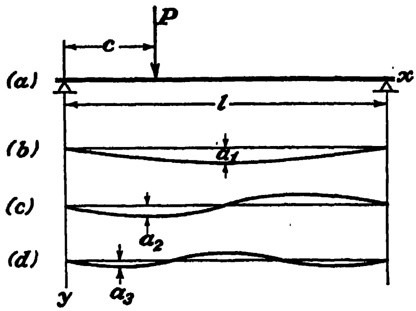


FIG. 166.

$$y = a_1 \sin\frac{\pi x}{l} + a_2 \sin\frac{2\pi x}{l} + a_3 \sin\frac{3\pi x}{l} + \cdots. \qquad (a)$$

Each term of this series and its second derivative vanish for $x = 0$ and for $x = l$, so that conditions at the simply supported ends of the beam are satisfied. It is seen that the deflection y for any value of x can be calculated if we know the amplitudes $a_1, a_2, \ldots$ of the sinusoidal curves and we take these quantities as generalized coordinates. To find the corresponding generalized forces, we consider the beam as a system of particles, infinite in number, on which the external force P together with the corresponding reactions at the supports and internal elastic forces are acting. All these forces have potential, and we can derive an expression for the potential energy of the system. We begin with the load P. The deflection of the beam at the point of application of this load is obtained by substituting $x = c$ in the series (a). Then the potential energy of the load P is

$$V_1 = -P\left(a_1 \sin\frac{\pi c}{l} + a_2 \sin\frac{2\pi c}{l} + a_3 \sin\frac{3\pi c}{l} + \cdots\right) = -P\sum a_n \sin\frac{n\pi c}{l}. \qquad (b)$$

The potential energy of the elastic forces will be found from the known formula for strain energy of bending (see page 168)

$$V_2 = \frac{EI}{2}\int_0^l \left(\frac{d^2y}{dx^2}\right)^2 dx, \qquad (c)$$

in which EI denotes the constant flexural rigidity of the beam. Substituting the series (a) for y, we obtain

$$V_2 = \frac{EI}{2}\int_0^l \left(a_1 \frac{\pi^2}{l^2}\sin\frac{\pi x}{l} + a_2\frac{4\pi^2}{l^2}\sin\frac{2\pi x}{l} + a_3\frac{9\pi^2}{l^2}\sin\frac{3\pi x}{l} + \cdots\right)^2 dx. \qquad (d)$$

Making the square of the series under the integral sign, we shall obtain two kinds of terms, namely: squares of the terms of the series, which have the form

$$a_n^2 \frac{n^4\pi^4}{l^4}\sin^2\frac{n\pi x}{l},$$

and double products of the form

$$2a_n a_m \frac{m^2 n^2 \pi^4}{l^4}\sin\frac{m\pi x}{l}\sin\frac{n\pi x}{l}.$$

Integrating these terms, we obtain

$$\int_0^l a_n^2 \frac{n^4\pi^4}{l^4} \sin^2 \frac{n\pi x}{l} dx = a_n^2 \frac{n^4\pi^4}{2l^3},$$
$$\int_0^l 2a_n a_m \frac{m^2 n^2 \pi^4}{l^4} \sin \frac{m\pi x}{l} \sin \frac{n\pi x}{l} dx = 0.$$

With these values of the integrals of the single terms, Eq. (*d*) becomes

$$V_2 = \frac{EI}{2}\left(\frac{a_1^2\pi^4}{2l^3} + \frac{a_2^2 2^4\pi^4}{2l^3} + \frac{a_3^2 3^4 \pi^4}{2l^3} + \cdots\right) = \frac{EI\pi^4}{4l^3}\sum n^4 a_n^2, \tag{e}$$

and the total potential energy of the system is

$$V = V_1 + V_2 = -P\sum a_n \sin\frac{n\pi c}{l} + \frac{EI\pi^4}{4l^3}\sum n^4 a_n^2. \tag{f}$$

The generalized force, corresponding to any generalized coordinate a_n, is now obtained from Eq. (130), and equations of equilibrium (132) take the form

$$-P\sin\frac{n\pi c}{l} + \frac{EI\pi^4}{2l^3} n^4 a_n = 0, \tag{g}$$

from which

$$a_n = \frac{2Pl^3}{EI\pi^4 n^4}\sin\frac{n\pi c}{l}. \tag{h}$$

In this way, each generalized coordinate can be calculated; and after substitution in the series (*a*), we obtain the equation for the deflection curve in the following form:

$$y = \frac{2Pl^3}{EI\pi^4}\sum \frac{1}{n^4}\sin\frac{n\pi c}{l}\sin\frac{n\pi x}{l}. \tag{i}$$

From this expression, the deflection at any cross section of the beam can be readily calculated.

Assume, for example, that we want the deflection under a load P applied at the middle of the beam. Then substituting $c = x = l/2$ into Eq. (*i*), we find

$$(y)_{x=\frac{l}{2}} = \frac{2Pl^3}{EI\pi^4}\left(1 + \frac{1}{3^4} + \frac{1}{5^4} + \cdots\right). \tag{j}$$

It is seen that in this case, all terms with even values of n vanish, since the deflection curve is symmetrical and all sinusoidal curves with an inflection point at the middle, as in Fig. 166*c*, cannot appear. We can also say that for each sinusoidal deflection with an inflection point at the middle, the deflection at the middle vanishes and the corresponding generalized coordinate, as can be seen from expression (*h*), is zero. The series (*j*) converges rapidly; and by taking only a few terms, we obtain deflections with a high degree of accuracy. Taking only the first term, we find

$$(y)_{x=\frac{l}{2}} \approx \frac{2Pl^3}{EI\pi^4} = \frac{Pl^3}{48.7EI}. \tag{k}$$

Expression (*k*) gives the deflection with an error of only about $1\frac{1}{2}$ per cent, which is a very satisfactory accuracy in practical applications. This means that a good approximation is obtained by taking, instead of the complete series (*a*), only its first term, which amounts to assuming that the deflection curve has the shape of the sine wave shown in Fig. 166*b*.

The last observation can be utilized in calculating deflections of beams the end conditions of which are different from those assumed in Fig. 166a. A rigorous solution of such a problem by the method of generalized coordinates brings us to a series more complicated than the series (a), but a satisfactory approximation can be obtained by assuming a suitable shape of the deflection curve and defining the deflections by one coordinate only. Take, for example, a beam with build-in ends and loaded at the middle (Fig. 167).

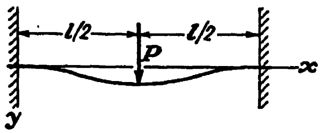


FIG. 167.

A suitable expression for the deflection curve satisfying the end conditions in this case is

$$y = a\left(1 - \cos\frac{2\pi x}{l}\right), \tag{l}$$

which gives zero deflection and zero slope at both ends of the beam. The deflection at the middle is

$$(y)_{x=\frac{l}{2}} = 2a.$$

The potential energy of the load P, then, is

$$V_1 = -2Pa, \tag{m}$$

and the potential energy of the elastic forces is

$$V_2 = \frac{EI}{2}\int_0^l \left(\frac{d^2y}{dx^2}\right)^2 dx = \frac{EI}{2}\int_0^l \left(\frac{4\pi^2}{l^2}a\cos\frac{2\pi x}{l}\right)^2 dx = \frac{4\pi^4 EI}{l^3}a^2. \tag{n}$$

Thus the equation of equilibrium will be

$$\frac{d}{da}(V_1 + V_2) = -2P + \frac{8\pi^4 EI}{l^3}a = 0,$$

from which we obtain

$$2a = \frac{2Pl^3}{4EI\pi^4} \tag{o}$$

for the deflection under the load P.

If the simply supported beam in Fig. 166a carries a uniformly distributed load of intensity w over the entire span, we can again obtain the deflection at any point by using the series (a). In such case, the potential energy of the elastic forces is still given by the series (e). To find the potential energy of the distributed load, we can use the series (b), replacing P by $w\,dc$ and integrating over the entire length l of the beam. Thus

$$V_1 = -w\int_0^l \left(a_1\sin\frac{\pi c}{l} + a_2\sin\frac{2\pi c}{l} + a_3\sin\frac{3\pi c}{l} + \cdots\right)dc$$
$$= -\frac{2wl}{\pi}\left(a_1 + \frac{a_3}{3} + \frac{a_5}{5} + \cdots\right). \tag{p}$$

Then the equations of equilibrium (132) becomes

$$-\frac{2wl}{\pi}\frac{1}{n} + \frac{EI\pi^4}{2l^3}n^4 a_n = 0, \tag{q}$$

where $n = 1, 3, 5, \ldots$. From this, we find

$$a_n = \frac{4wl^4}{EI\pi^5 n^5}. \tag{r}$$

Substituting the coefficients (r) into the series (a), we obtain, for the equation of the deflection curve,

$$y = \frac{4wl^4}{EI\pi^5} \sum \frac{1}{n^5} \sin \frac{n\pi x}{l}, \qquad (s)$$

in which $n = 1, 3, 5, \ldots$.

PROBLEMS

107. Using the series (a), find the deflection curve for a simply supported beam carrying two equal loads P applied at equal distances c from the two ends of the beam.

Ans. $y = \dfrac{4Pl^3}{EI\pi^4} \sum \dfrac{1}{n^4} \sin \dfrac{n\pi c}{l} \sin \dfrac{n\pi x}{l}.$

108. A cantilever beam of length l carries a load P at its free end as shown in Fig. 168. Assuming, for the deflection curve, the expression

$$y = a\left(1 - \cos \frac{\pi x}{2l}\right),$$

find the deflection under the load P.

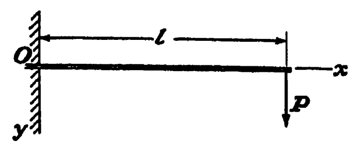


FIG. 168.

109. Using the data of the preceding problem, find the deflection at the free end of a uniformly loaded cantilever beam.

110. Using the series (a), find the expression for the deflection curve of a simply supported beam that carries a uniformly distributed load of intensity w over one-half of the span.

27. D'Alembert Principle.—The methods discussed in the preceding two articles for solving problems of statics can be applied also in deriving equations of motion. By using D'Alembert's principle,[1] equations of dynamics can be written in the same manner as the equations of statics. It is necessary only to add to the given external forces the inertia forces for the various particles of the system. Thus, for any particle i of a system, the equations of motion, in rectangular coordinates, can be written in the following form:

$$\left.\begin{aligned} X_i + X_i' - m_i\ddot{x}_i &= 0, \\ Y_i + Y_i' - m_i\ddot{y}_i &= 0, \\ Z_i + Z_i' - m_i\ddot{z}_i &= 0. \end{aligned}\right\} \qquad (a)$$

In these expressions, X_i, Y_i, Z_i are the components of the resultant of all active forces applied to the particle, X_i', Y_i', Z_i' the components of the resultant of forces of constraint for that particle, and $-m_i\ddot{x}_i$, $-m_i\ddot{y}_i$, $-m_i\ddot{z}_i$ the components of its inertia force. We see that Eqs. (a) have the same form as equations of statics of a particle in rectangular coordinates. Such equations can be written for each particle of the system.

To simplify the problem and eliminate the unknown forces of constraint, we shall generally use, instead of Eqs. (a), equilibrium equations derived by the principle of virtual displacements. Giving to any particle a virtual displacement with components δx_i, δy_i, δz_i, we find, for virtual

[1] See the authors' "Engineering Mechanics," 2d ed., p. 311.

work, the expression

$$(X_i + X_i' - m_i\ddot{x}_i)\delta x_i + (Y_i + Y_i' - m_i\ddot{y}_i)\delta y_i + (Z_i + Z_i' - m_i\ddot{z}_i)\delta z_i. \quad (b)$$

Similar expressions can be written for each particle of the system; and by summing them up, the total virtual work will be obtained. We assume now that the system has ideal constraints; *i.e.*, there is no friction in hinges or guides, and connecting bars are absolutely rigid (see page 197). In such case, for each virtual displacement of the system, the work of all forces of constraint vanishes; and after summation of expressions (b), we obtain

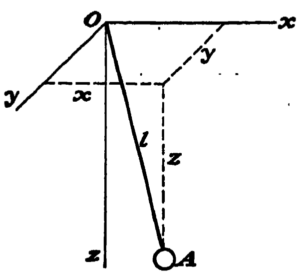


FIG. 169.

$$\Sigma[(X_i - m_i\ddot{x}_i)\delta x_i + (Y_i - m_i\ddot{y}_i)\delta y_i + (Z_i - m_i\ddot{z}_i)\delta z_i]. \quad (c)$$

In this expression, the summation must include all particles of the system and the assumed displacements must be compatible with the constraints of the system; *i.e.* the increments δx_i, δy_i, δz_i of the coordinates must be such that all equations of constraint are satisfied.

Take, for example, the case of a spherical pendulum as shown in Fig. 169. The coordinates of the particle A must satisfy the equation of constraint

$$x^2 + y^2 + z^2 = l^2, \quad (d)$$

and the virtual displacements δx, δy, δz must be compatible with this equation, which requires that

$$(x + \delta x)^2 + (y + \delta y)^2 + (z + \delta z)^2 = l^2. \quad (e)$$

Neglecting the squares of small quantities, we obtain, from Eqs. (d) and (e) together,

$$x\,\delta x + y\,\delta y + z\,\delta z = 0. \quad (f)$$

This equation shows that a virtual displacement of the particle A is perpendicular to OA. Thus any small displacement of the particle on the surface of a sphere of radius l and center at O can be considered as a virtual displacement. From the three components δx, δy, δz of such a virtual displacement, two can be taken arbitrarily and the third will be calculated from Eq. (f). We have a system with two degrees of freedom.

Returning to the general case and remembering that the inertia forces applied to the various particles equilibrate the system, we conclude that expression (c) for virtual work must vanish for each assumed virtual

displacement of the system. Thus

$$\Sigma[(X_i - m_i\ddot{x}_i)\delta x_i + (Y_i - m_i\ddot{y}_i)\delta y + (Z_i - m_i\ddot{z}_i)\delta z_i] = 0. \quad (133)$$

If the system has n particles, the $3n$ coordinates of which are connected by k equations of constraint, we shall have $3n - k$ degrees of freedom and the same number of different virtual displacements for which $(3n - k)$ equations like Eq. (133) can be written. These equations together with the k equations of constraint are sufficient for determining the $3n$ coordinates of all particles of the system.

If there are no constraints and the particles are entirely free, the displacements δx_i, δy_i, δz_i can be taken arbitrarily. For example, we can take all of them except δx_i equal to zero. Then all terms in Eq. (133), except that containing δx_i, vanish, and we obtain

$$(X_i - m_i\ddot{x}_i)\delta x_i = 0.$$

Since δx_i is not zero, the expression in the parenthesis must be zero and we obtain

$$m_i\ddot{x}_i = X_i,$$

representing the known equations of motion of free particles.

We derived the equation of virtual work (133) in rectangular coordinates; but in the case of systems with constraints, generalized coordinates can be used more advantageously. Equations of statics, in such case, are given by Eqs. (131), and to obtain from them equations of motion, we have only to add to the given forces the inertia forces for the various particles. Proceeding then, as explained in Art. 24, we obtain, for generalized coordinates $q_1, q_2, \ldots q_n$, the corresponding generalized forces $Q_1 + S_1, Q_2 + S_2, \ldots Q_n + S_n$, in which Q_i denotes that part of any generalized force obtained from given external forces and S_i that part obtained from the inertia forces. The equations of motion then are

$$Q_1 + S_1 = 0, \qquad Q_2 + S_2 = 0, \qquad \cdots \qquad Q_n + S_n = 0. \quad (134)$$

The number of these equations is equal to the number of generalized coordinates, specifying the configuration of the system, and by their integration the motion of the system is completely defined.

Since Eqs. (134) are written in the same way as equations of statics, there will be no difficulty, in any particular case, to write the necessary equations of motion. Thus D'Alembert's principle in conjunction with the principle of virtual displacements gives a general method of attack for dynamical problems involving systems with constraints.

Take, for example, a theoretical double pendulum, (Fig. 170). This system consists of two particles, the four coordinates of which are connected by two equations of constraint [see Eqs. (*c*), page 191]. The

system has two degrees of freedom, and we have to write two equations of motion. Using D'Alembert's principle, we add to the given external forces W_1 and W_2 the inertia forces shown in Fig. 170. Applying now to the obtained system of forces the principle of virtual displacements, we observe that the configuration of the system is completely defined by the angles ψ and ϕ and two independent virtual displacements will be obtained by giving to these angles the increments $\delta\psi$ and $\delta\phi$, respectively. During the displacement $\delta\psi$, the particle A remains stationary and the virtual work is obtained by multiplying by $\delta\psi$ the moment, with respect to point A, of all forces applied to the particle B. This moment, then, is the generalized force corresponding to ψ. Equating it to zero, we obtain

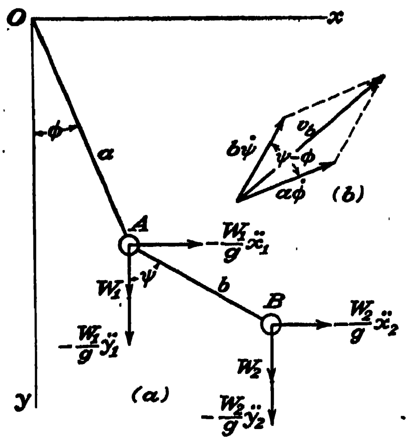


FIG. 170.

$$\left(W_2 - \frac{W_2}{g}\ddot{y}_2\right) b \sin\psi + \frac{W_2}{g}\ddot{x}_2 b \cos\psi = 0. \qquad (g)$$

Similarly, corresponding to the virtual displacement $\delta\phi$, we must equate to zero the moment of all forces with respect to point O, which gives

$$\left(W_2 - \frac{W_2}{g}\ddot{y}_2\right)(b\sin\psi + a\sin\phi) + \frac{W_2}{g}\ddot{x}_2(b\cos\psi + a\cos\phi) + \left(W_1 - \frac{W_1}{g}\ddot{y}_1\right) a\sin\phi + \frac{W_1}{g}\ddot{x}_1 a\cos\phi = 0. \qquad (h)$$

To obtain the equations of motion in final form, we have only to subsitute for the rectangular coordinates their expressions

$$\left.\begin{aligned} x_1 &= a\sin\phi, & y_1 &= a\cos\phi, \\ x_2 &= a\sin\phi + b\sin\psi, & y_2 &= a\cos\phi + b\cos\psi. \end{aligned}\right\} \qquad (i)$$

The results of such substitution will be involved, but the problem is greatly simplified if we limit our consideration to small oscillations. Considering ϕ and ψ and their derivatives with respect to time as small quantities and neglecting terms containing products and powers of small quantities, we obtain, from Eqs. (i),

$$\ddot{x}_1 \approx a\ddot{\phi}, \qquad \ddot{x}_2 \approx a\ddot{\phi} + b\ddot{\psi}, \qquad \ddot{y}_1 \approx 0, \qquad \ddot{y}_2 \approx 0.$$

Substituting these approximations into Eqs. (g) and (h), we obtain

$$W_2 b\psi + \frac{W_2}{g} b(a\ddot{\phi} + b\ddot{\psi}) = 0, \qquad (j)$$

$$W_2(b\psi + a\phi) + W_1 a\phi + \frac{W_2}{g}(a + b)(a\ddot{\phi} + b\ddot{\psi}) + \frac{W_1}{g} a^2\ddot{\phi} = 0.$$

The second of these equations can be somewhat simplified by subtraction of the first, which gives

$$(W_2 + W_1)a\phi + \frac{W_2}{g}(a^2\ddot{\phi} + ab\ddot{\psi}) + \frac{W_1}{g} a^2\ddot{\phi} = 0. \qquad (k)$$

Equations (j) and (k) are the desired equations of motion for small oscillations of the theoretical double pendulum. The solution of such equations will be discussed later.

PROBLEMS

111. Using D'Alembert's principle, write equations of motion of the system with two disks shown in Fig. 159, p. 199.

Ans. $I_2\ddot{\psi} = -(\psi - \phi)\frac{GJ}{b}$, $I_1\ddot{\phi} = (\psi - \phi)\frac{GJ}{b} - \phi\frac{GJ}{a}$.

112. Using D'Alembert's principle, write equations of motion of the two particles shown in Fig. 160, page 200.

Ans. $\frac{W_2}{g}\ddot{y}_b = -k_2(y_b - y_a)$, $\frac{W_1}{g}\ddot{y}_a = k_2(y_b - y_a) - k_1 y_a$.

113. Derive the equations of motion for small oscillations of the double physical pendulum shown in Fig. 171. Points C and C_1 are the centers of gravity of the two

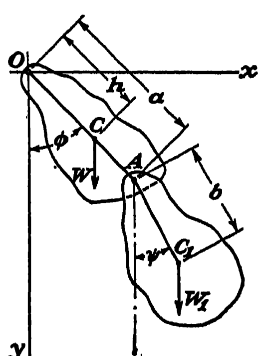


FIG. 171.

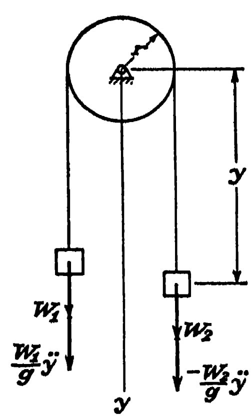


FIG. 172.

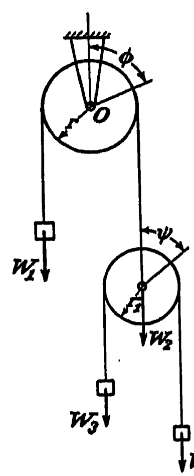


FIG. 173.

bodies; i and i_1 their radii of gyration with respect to O and A, respectively; and W and W_1, their weights.

Ans. $\left(\frac{W}{g} i^2 + \frac{W_1}{g} a^2\right)\ddot{\phi} + (Wh + W_1 a)\phi + \frac{W_1}{g} ab\ddot{\psi} = 0,$

$ab\ddot{\phi} + i_1^2\ddot{\psi} + gb\psi = 0.$

114. Derive the equations of motion of the blocks W_1 and W_2 of the Atwood's machine shown in Fig. 172, neglecting the mass of the pulley and of the string.

Ans. $(W_1 + W_2)\ddot{y} = g(W_2 - W_1)$.

115. Derive the equations of motion for the system of pulleys and blocks shown in Fig. 173. Neglect the moments of inertia of the pulleys and the mass of the string and take the angles ϕ and ψ as generalized coordinates.

Ans.
$$\frac{r}{g}(W_1 + W_2 + W_3 + W_4)\ddot{\phi} - \frac{r_1}{g}(W_3 - W_4)\ddot{\psi} = -W_1 + W_2 + W_3 + W_4,$$
$$\frac{r}{g}(W_3 - W_4)\ddot{\phi} - \frac{r_1}{g}(W_4 + W_3)\ddot{\psi} = W_3 - W_4.$$

28. Lagrangian Equations.—Equation (133), derived on the basis of D'Alembert's principle, can be expressed more simply by using the notion of kinetic energy together with generalized coordinates. For this purpose, we transfer the terms, containing forces to the right-hand side and write

$$\Sigma m_i(\ddot{x}_i\delta x_i + \ddot{y}_i\delta y_i + \ddot{z}_i\delta z_i) = \Sigma(X_i\delta x_i + Y_i\delta y_i + Z_i\delta z_i). \qquad (a)$$

Now let $q_1, q_2, \ldots q_n$ denote the generalized coordinates of a system and $Q_1, Q_2, \ldots Q_n$ the corresponding generalized forces. Assume also that equations like Eqs. (*i*) of the preceding article contain only coordinates[1] $q_1, q_2, \ldots q_n$, *i.e.*,

$$\left.\begin{aligned} x_i &= F_i(q_1,q_2, \ldots q_n), \\ y_i &= G_i(q_1,q_2, \ldots q_n), \\ z_i &= H_i(q_1,q_2, \ldots q_n). \end{aligned}\right\} \qquad (b)$$

We have a system with n degrees of freedom and can write n equations of motion (*a*) considering n different virtual displacements of the system, corresponding to the increments $\delta q_1, \delta q_2, \ldots \delta q_n$ of the generalized coordinates. Let us consider the virtual displacement corresponding to an increment δq_s of the coordinate q_s. The corresponding changes in rectangular coordinates of the particles of the system can be calculated from the equations

$$\delta x_i = \frac{\partial x_i}{\partial q_s}\delta q_s, \qquad \delta y_i = \frac{\partial y_i}{\partial q_s}\delta q_s, \qquad \delta z_i = \frac{\partial z_i}{\partial q_s}\delta q_s, \qquad (c)$$

in which the symbol $\partial/\partial q_s$ denotes the partial derivative with respect to the chosen coordinate q_s and x_i, y_i, z_i are given by Eqs. (*b*). The virtual work, corresponding to the displacement δq_s is, by definition, $Q_s\delta q_s$, and Eq. (*a*) may be written in the form[2]

$$\sum m\left(\ddot{x}\frac{\partial x}{\partial q_s} + \ddot{y}\frac{\partial y}{\partial q_s} + \ddot{z}\frac{\partial z}{\partial q_s}\right)\delta q_s = Q_s\delta q_s.$$

[1] The cases in which they contain also time t will be discussed in Art. 29.

[2] To simplify writing, we omit the subscript i in rectangular coordinates of the particles.

Canceling δq_s, we obtain

$$\sum m\left(\ddot{x}\frac{\partial x}{\partial q_s} + \ddot{y}\frac{\partial y}{\partial q_s} + \ddot{z}\frac{\partial z}{\partial q_s}\right) = Q_s. \tag{d}$$

To transform the left-hand side of Eq. (d) to generalized coordinates, we use the expression for kinetic energy

$$T = \tfrac{1}{2}\Sigma m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$$

and make the partial derivatives

$$\frac{\partial T}{\partial \dot{q}_s} = \sum m\left(\dot{x}\frac{\partial \dot{x}}{\partial \dot{q}_s} + \dot{y}\frac{\partial \dot{y}}{\partial \dot{q}_s} + \dot{z}\frac{\partial \dot{z}}{\partial \dot{q}_s}\right), \tag{e}$$

$$\frac{\partial T}{\partial q_s} = \sum m\left(\dot{x}\frac{\partial \dot{x}}{\partial q_s} + \dot{y}\frac{\partial \dot{y}}{\partial q_s} + \dot{z}\frac{\partial \dot{z}}{\partial q_s}\right). \tag{f}$$

Returning now to Eqs. (b), we obtain, by differentiation,

$$\dot{x} = \frac{dx}{dt} = \frac{\partial x}{\partial q_1}\dot{q}_1 + \frac{\partial x}{\partial q_2}\dot{q}_2 + \cdots + \frac{\partial x}{\partial q_n}\dot{q}_n, \tag{g}$$

$$\frac{\partial \dot{x}}{\partial q_s} = \frac{\partial^2 x}{\partial q_1 \partial q_s}\dot{q}_1 + \frac{\partial^2 x}{\partial q_2 \partial q_s}\dot{q}_2 + \cdots + \frac{\partial^2 x}{\partial q_n \partial q_s}\dot{q}_n, \tag{h}$$

$$\frac{d}{dt}\left(\frac{\partial x}{\partial q_s}\right) = \frac{\partial^2 x}{\partial q_1 \partial q_s}\dot{q}_1 + \frac{\partial^2 x}{\partial q_2 \partial q_s}\dot{q}_2 + \cdots + \frac{\partial^2 x}{\partial q_n \partial q_s}\dot{q}_n. \tag{i}$$

Similar equations can be written also for coordinates y and z. In all these equations, the derivatives with respect to t are total derivatives, and those with respect to coordinates are partial derivatives. From Eq. (g), we conclude that

$$\frac{\partial \dot{x}}{\partial \dot{q}_s} = \frac{\partial x}{\partial q_s}; \tag{j}$$

and from comparison of Eqs. (h) and (i), we obtain

$$\frac{\partial \dot{x}}{\partial q_s} = \frac{d}{dt}\left(\frac{\partial x}{\partial q_s}\right). \tag{k}$$

Similar conclusions can be obtained for y and z coordinates, and Eqs. (e) and (f) can be rewritten in the following form:

$$\frac{\partial T}{\partial \dot{q}_s} = \sum m\left(\dot{x}\frac{\partial x}{\partial q_s} + \dot{y}\frac{\partial y}{\partial q_s} + \dot{z}\frac{\partial z}{\partial q_s}\right), \tag{l}$$

$$\frac{\partial T}{\partial q_s} = \sum m\left(\dot{x}\frac{d}{dt}\frac{\partial x}{\partial q_s} + \dot{y}\frac{d}{dt}\frac{\partial y}{\partial q_s} + \dot{z}\frac{d}{dt}\frac{\partial z}{\partial q_s}\right). \tag{m}$$

Returning now to the differential equation (d) and observing that

$$\ddot{x}\frac{\partial x}{\partial q_s} + \ddot{y}\frac{\partial y}{\partial q_s} + \ddot{z}\frac{\partial z}{\partial q_s} = \frac{d}{dt}\left(\dot{x}\frac{\partial x}{\partial q_s} + \dot{y}\frac{\partial y}{\partial q_s} + \dot{z}\frac{\partial z}{\partial q_s}\right) - \left(\dot{x}\frac{d}{dt}\frac{\partial x}{\partial q_s} + \dot{y}\frac{d}{dt}\frac{\partial y}{\partial q_s} + \dot{z}\frac{d}{dt}\frac{\partial z}{\partial q_s}\right),$$

we rewrite this equation in the form

$$\frac{d}{dt}\left[\sum m\left(\dot{x}\frac{\partial x}{\partial q_s} + \dot{y}\frac{\partial y}{\partial q_s} + \dot{z}\frac{\partial z}{\partial q_s}\right)\right] - \sum m\left(\dot{x}\frac{d}{dt}\frac{\partial x}{\partial q_s} + \dot{y}\frac{d}{dt}\frac{\partial y}{\partial q_s} + \dot{z}\frac{d}{dt}\frac{\partial z}{\partial q_s}\right) = Q_s.$$

Using Eqs. (l) and (m), this reduces to

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_s} - \frac{\partial T}{\partial q_s} = Q_s. \tag{135}$$

This equation holds for each generalized coordinate, so that we have always as many equations as there are independent coordinates specifying the configuration of a system. These are the famous Lagrangian equations, which are so widely used in physics and engineering. To write them for any particular system, we need only the expression for kinetic energy T of the system in generalized coordinates. Having this, the left-hand sides of Eqs. (135) are obtained by simple differentiation. On the right-hand sides, we have the generalized forces which in each particular case can be calculated as shown in Art. 24.

The advantage of the Lagrangian equations lies in the fact that equations of motion for any system with ideal constraints can be written, so to say, automatically by rule, without any additional consideration.[1] Naturally, this reliance on formulas alone has its negative side, since, by such procedure, the physical picture of the actual movement of the system due to given forces is somewhat obscured and simplifications that might be obtained by using the general principles discussed previously are likely to remain unnoticed.

If the given forces have potential, the generalized forces can be found from Eq. (130) and the Lagrangian equations become

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_s} - \frac{\partial T}{\partial q_s} + \frac{\partial V}{\partial q_s} = 0. \tag{136}$$

If there are two kinds of forces, one that has potential and can be

[1] E. Mach, in his work "Mechanics in Its Development," considers that the purpose of every science is to describe and represent certain groups of facts and relations between them in such a manner that it will take the minimum of mental effort to understand them. From this point of view of economy of thought, Lagrangian equations represent a great contribution to the science of mechanics. They do not so much promote insight into mechanical processes as the practical mastery of them.

derived from Eq. (130) and the other that has no potential, the Lagrangian equations can be written in the form

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_s} - \frac{\partial T}{\partial q_s} + \frac{\partial V}{\partial q_s} = Q_s, \tag{137}$$

in which Q_s is the generalized force corresponding to that part of the given forces which have no potential.

As a first example of the use of Lagrangian equations, we consider again the free oscillations of the double pendulum in Fig. 170a and begin with a calculation of the kinetic energy of the system. The velocity of the particle A is $v_a = a\dot{\phi}$. To find the velocity v_b of the particle B, we observe that it moves together with the particle A and in addition rotates with respect to A so that its total velocity is obtained by geometric addition of these two velocities as shown in Fig. 170b. From this figure, we find

$$v_b^2 = a^2\dot{\phi}^2 + b^2\dot{\psi}^2 + 2ab\dot{\phi}\dot{\psi}\cos(\psi - \phi),$$

and the kinetic energy of the system is

$$T = \frac{1}{2}\frac{W_1}{g}a^2\dot{\phi}^2 + \frac{1}{2}\frac{W_2}{g}[a^2\dot{\phi}^2 + b^2\dot{\psi}^2 + 2ab\dot{\phi}\dot{\psi}\cos(\psi - \phi)].$$

Using this expression in Eq. (135) and observing that the generalized forces corresponding to the coordinates ϕ and ψ are given by Eqs. (d) and (c) of Art. 24, we obtain

$$\frac{d}{dt}\left[\frac{1}{g}(W_1 + W_2)a^2\dot{\phi} + \frac{1}{g}W_2ab\dot{\psi}\cos(\psi - \phi)\right] - \frac{1}{g}W_2\,ab\dot{\phi}\dot{\psi}\sin(\psi - \phi) = -(W_1 + W_2)a\sin\phi,$$

$$\frac{d}{dt}\left\{\frac{1}{g}W_2\,[b^2\dot{\psi} + ab\dot{\phi}\cos(\psi - \phi)]\right\} + \frac{1}{g}W_2\,ab\dot{\phi}\dot{\psi}\sin(\psi - \phi) = -W_2\,b\sin\psi.$$

These equations are greatly simplified in the case of small oscillations. Assuming that ϕ and ψ, together with their derivatives with respect to time, are always small, we can take

$$\sin\phi \approx \phi, \qquad \sin\psi \approx \psi, \qquad \sin(\psi - \phi) \approx \psi - \phi, \qquad \cos(\psi - \phi) \approx 1.$$

Then neglecting all terms containing squares or products of small quantities, the above equations reduce to

$$\left.\begin{aligned} \frac{1}{g}(W_1 + W_2)a^2\ddot{\phi} + \frac{1}{g}W_2ab\ddot{\psi} &= -(W_1 + W_2)a\phi, \\ \frac{1}{g}W_2(b^2\ddot{\psi} + ab\ddot{\phi}) &= -W_2b\psi, \end{aligned}\right\} \tag{n}$$

which coincide with Eqs. (k) and (j) of the preceding article.

Since, in this case, the forces have potential energy

$$V = W_1a(1 - \cos\phi) + W_2[a(1 - \cos\phi) + b(1 - \cos\psi)],$$

we can also use Lagrangian equations in the form (136), which brings us to the same result.[1]

Assume now that in addition to gravity forces W_1 and W_2, there is a horizontal pulsating force $P \sin \omega t$ applied to the particle B in Fig. 170a. This force has no potential, and we must use Lagrangian equations in the form (137). The generalized forces corresponding to $P \sin \omega t$ are

$$(a \cos\phi + b \cos\psi)P \sin \omega t, \qquad b \cos\psi \cdot P \sin \omega t;$$

and for small oscillations, Eqs. (137) give

$$\left.\begin{aligned} \frac{1}{g}(W_1 + W_2)a^2\ddot{\phi} + \frac{1}{g}W_2ab\ddot{\psi} &= -(W_1 + W_2)a\phi + (a + b)P \sin \omega t, \\ \frac{1}{g}W_2(b^2\ddot{\psi} + ab\ddot{\phi}) &= -W_2b\psi + bP \sin \omega t. \end{aligned}\right\} \qquad (n')$$

The solution of these equations will be discussed later.

As a second example, let us consider the equations of motion of the spherical pendulum shown in Fig. 174. The position of the particle A on the sphere is completely defined by the angles ϕ and θ which we take for generalized coordinates. Corresponding to an increment $\delta\phi$ of the coordinate ϕ, we make a small displacement of the particle along the arc of the parallel circle and, corresponding to $\delta\theta$, a displacement along the meridian. Considering the work of the gravity force W on these displacements, we conclude that the generalized force corresponding to ϕ vanishes while that corresponding to θ is equal to $-Wl \sin \theta$. The components of the velocity of the particle A in the directions of the tangents to the parallel circle and to the meridian, respectively, are

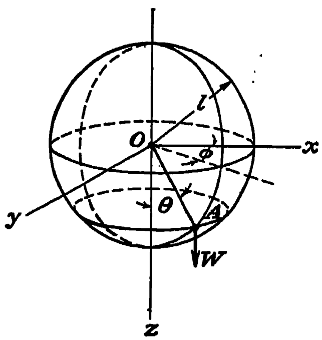


FIG. 174.

$$l\dot{\phi} \sin \theta \qquad \text{and} \qquad l\dot{\theta},$$

and the kinetic energy of the system is

$$T = \frac{W}{2g}[(l\dot{\theta})^2 + (l\dot{\phi} \sin \theta)^2].$$

[1] We note that to get the equations of motion with the accuracy of small quantities of the first order, the expressions for kinetic and potential energy must be derived with the accuracy up to small quantities of the second order.

Lagrangian equations (135) then give the equations of motion

$$\left.\begin{aligned} \ddot{\theta} - \dot{\phi}^2 \sin\theta\cos\theta &= -\frac{g}{l}\sin\theta, \\ \frac{d}{dt}(\dot{\phi}\sin^2\theta) &= 0. \end{aligned}\right\} \tag{o}$$

Let us consider now several particular cases.

Case I. The particle moves only along the meridian, $\dot{\phi} = 0$, and we obtain, from the first of Eqs. (o),

$$\ddot{\theta} + \frac{g}{l}\sin\theta = 0. \tag{p}$$

This is the known equation of motion for a theoretical pendulum in a vertical plane.

Case II. The angle θ remains constant during motion and equal to α. Then, as we see from the second of Eqs. (o), $\dot{\phi}$ must also be constant and the first equation gives

$$\dot{\phi}^2 = \omega^2 = \frac{g}{l\cos\alpha}. \tag{q}$$

This is the known equation of motion of a conical pendulum.

Case III. We assume now that the steady motion of the conical pendulum is slightly disturbed so that there are small transverse oscillations of the particle A about its horizontal circular path. For such case, we take

$$\theta = \alpha + \xi,$$

where ξ denotes small fluctuations in the angle θ during motion. Considering ξ as a small quantity and neglecting terms containing ξ in powers higher than the first, we have

$$\sin\theta = \sin\alpha + \xi\cos\alpha, \qquad \cos\theta = \cos\alpha - \xi\sin\alpha.$$

Now integrating the second of Eqs. (o) and denoting the constant of integration by C, we find

$$\dot{\phi}^2 = \left(\frac{C}{\sin^2\theta}\right)^2 \approx \left[\frac{C}{\sin^2\alpha(1 + 2\xi\cot\alpha)}\right]^2 \approx \omega^2(1 - 4\xi\cot\alpha),$$

Substituting this approximation for $\dot{\phi}^2$ into the first of Eqs. (o) gives

$$\ddot{\xi} - \omega^2\sin\alpha\left[\cos\alpha - \xi\left(\frac{3\cos^2\alpha}{\sin\alpha} + \sin\alpha\right)\right] = -\frac{g}{l}(\sin\alpha + \xi\cos\alpha). \tag{r}$$

Finally, we note from Eq. (q) above, that

$$\frac{g}{l} = \omega^2\cos\alpha;$$

and with a little trigonometric manipulation, Eq. (r) can be reduced to

$$\ddot{\xi} + (1 + 3\cos^2\alpha)\omega^2\xi = 0. \tag{s}$$

This indicates that ξ represents harmonic oscillations having the period

$$\tau = \frac{2\pi}{\omega\sqrt{1 + 3\cos^2\alpha}}.$$

When α is small, $\cos\alpha$ approaches unity and the period τ approaches the value π/ω, which means that approximately two oscillations occur for each revolution of the conical pendulum. When α approaches $\pi/2$, $\cos\alpha$ approaches zero and the period τ approaches the value $2\pi/\omega$, *i.e.*, there is approximately one oscillation per revolution.

As a third example, we shall use Lagrangian equations to investigate small oscillations of the pendulum in Fig. 156, page 196. This is a system with one degree of freedom, and we take as generalized coordinate the angle ψ. This angle and the corresponding angular velocity $\dot{\psi}$ we shall consider as small quantities. Neglecting the masses of the bars, we have to calculate only the kinetic energy of the particle W. The velocity of this particle consists of two parts: (1) the velocity due to rotation of the bar OC with respect to O and (2) the velocity along this bar due to sliding in the sleeve O. Denoting by x the variable distance OB, we obtain, for the square of the total velocity,

$$v^2 = \dot{x}^2 + (l + x)^2\dot{\psi}^2,$$

and the kinetic energy is

$$T = \frac{W}{2g}[\dot{x}^2 + (l + x)^2\dot{\psi}^2]. \tag{t}$$

It remains to express x as a function of the chosen coordinate ψ. Considering the triangle ABO, we may write

$$\left.\begin{aligned} x\cos\psi + r\cos\phi &= h, \\ x\sin\psi &= r\sin\phi. \end{aligned}\right\} \tag{u}$$

Then treating ϕ and ψ as small angles, we put

$$\cos\psi \approx 1 - \tfrac{1}{2}\psi^2, \qquad \cos\phi \approx 1 - \tfrac{1}{2}\phi^2, \qquad \sin\psi \approx \psi, \qquad \sin\phi \approx \phi.$$

With these approximations, the first of Eqs. (u) becomes

$$x - \tfrac{1}{2}x\psi^2 + r - \tfrac{1}{2}r\phi^2 = h. \tag{v}$$

It is seen that x differs from $h - r$ only by small quantities containing ϕ^2 and ψ^2 as factors; hence, neglecting small quantities above the second order, we can substitute the quantity $h - r$ for x in the second term of Eq. (v) and write

$$x = h - r + \tfrac{1}{2}(h - r)\psi^2 + \tfrac{1}{2}r\phi^2. \tag{w}$$

From the second of Eqs. (u), we have

$$\phi \approx \frac{x}{r}\psi \approx \frac{h-r}{r}\psi.$$

Substituting in Eq. (w), we obtain

$$x = h - r + \frac{h}{2r}(h - r)\psi^2,$$

and the corresponding velocity is

$$\dot{x} = \frac{h}{r}(h - r)\psi\dot{\psi}. \tag{x}$$

Since this expression contains the product of two small quantities, its square is small in comparison with the second term in the brackets of Eq. (t) and can be neglected. In the term $(l + x)^2\dot{\psi}^2$, we can again neglect small quantities of higher order and substitute $h - r$ for x. In this way, we obtain the following simplified expression for kinetic energy

$$T = \frac{W}{2g}(l + h - r)^2\dot{\psi}^2. \tag{y}$$

The potential energy of the weight W with respect to its lowest position is

$$V = W[r\cos\phi - l\cos\psi - (r - l)].$$

Using the same approximations for $\cos\phi$ and $\cos\psi$ as above and keeping only small quantities of second order, this is readily reduced to

$$V = W\left[l - \frac{(h-r)^2}{r}\right]\frac{\psi^2}{2}. \tag{z}$$

Substituting the expressions for T and V into Eq. (136), we obtain

$$\frac{1}{g}(l + h - r)^2\ddot{\psi} + \left[l - \frac{(h-r)^2}{r}\right]\psi = 0.$$

Assuming that the quantity within the brackets of the second term of this equation is positive, we have a simple harmonic motion; *i.e.*,

$$\ddot{\psi} + p^2\psi = 0,$$

wherein

$$p^2 = \frac{[l - (h - r)^2/r]g}{(l + h - r)^2}.$$

By a proper dimensioning, we can make p as small as we like and obtain a very low natural frequency of vibration.

PROBLEMS

116. Using Lagrangian equations in the form (135), write the equations of motion for the system in Fig. 173, page 211.

117. Using Lagrangian equations in the form (136), write the equations of motion for the system shown in Fig. 175, consisting of two circular disks on the ends of a horizontal circular steel shaft of length l and diameter d. Neglect friction in the bearings A and B.

Ans. $I_1\ddot{\phi}_1 - \frac{GJ}{l}(\phi_2 - \phi_1) = 0;\ I_2\ddot{\phi}_2 + \frac{GJ}{l}(\phi_2 - \phi_1) = 0.$

118. In Fig. 176, the block A of weight W_1 can slide freely on a horizontal rod and the pendulum AB can rotate freely about A. Taking x_a and ϕ as generalized coordinates and using Lagrangian equations (136), write the equations of motion for the system. Neglect the mass of the rigid bar AB, and assume that x_a and ϕ together with their derivatives with respect to time are small quantities.

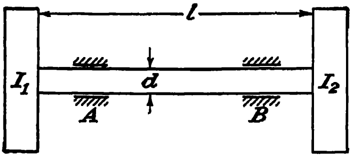


FIG. 175.

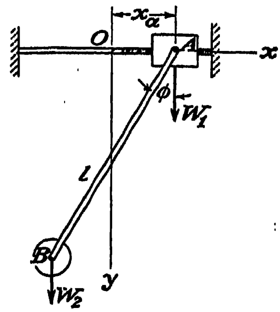


FIG. 176.

Ans. $(W_1 + W_2)\ddot{x}_a - W_2 l\ddot{\phi} = 0,\ l\ddot{\phi} - \ddot{x}_a + g\phi = 0.$

119. Show that the two equations of motion for the system in the preceding problem can be reduced to

$$\ddot{\phi} + \frac{W_1 + W_2}{W_1}\frac{g}{l}\phi = 0.$$

120. Two mass particles m_1 and m_2, connected by a flexible but inextensible string of length l, are arranged as shown in Fig. 177. The particle A remains on a smooth horizontal plane, the string passes through a small hole at O, and the particle B hangs vertically. Using Lagrangian equations and taking r and θ as generalized coordinates, write the equations of motion for the system.

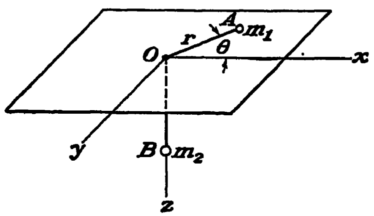


FIG. 177.

Ans. $d/dt\,(r^2\dot{\theta}) = 0;$

$(m_1 + m_2)\ddot{r} - m_1 r\dot{\theta}^2 = -m_2 g.$

121. If the particle A in the preceding problem rotates about point O with constant angular velocity $\dot{\theta} = \omega$, the radius r also remains constant, say $r = a$, and we have a steady state of motion in which the centrifugal force of the one particle just balances the weight of the other. Assuming then that a small disturbance is given to the particle A so that $r = a + \xi$, where ξ is a small quantity, show that the system will perform small simple harmonic oscillations with the period

$$\tau = \frac{2\pi}{\omega\sqrt{3m_1/(m_1 + m_2)}}.$$

122. A flywheel, rotating in a horizontal plane about its center O, carries a particle of mass m that can slide freely along one spoke and is attached to the center of the wheel by a spring of characteristic k as shown in Fig. 178. The moment of inertia

of the wheel about its axis of rotation is I, and the unstrained length of the spring is a. Taking, as generalized coordinates, the angle of rotation θ of the wheel and the elongation x of the spring, derive the equations of motion of the system.

Ans. $d/dt[I\dot{\theta} + m(a + x)^2\dot{\theta}] = 0$; $m\ddot{x} - m(a + x)\dot{\theta}^2 + kx = 0$.

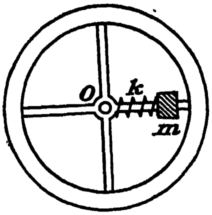


FIG. 178.

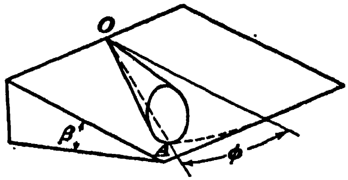


FIG. 179.

123. If the wheel in the preceding problem rotates with constant angular velocity $\dot{\theta} = \omega$, the elongation x of the spring also remains constant, say $x = c$, and we have a steady state in which the centrifugal force of the mass m is just balanced by the tension in the spring. Assuming now that the mass m is slightly disturbed so that $x = c + \xi$, where ξ is a small quantity, show that it will perform simple harmonic vibrations defined by the equation

$$m\ddot{\xi} - \left[m\omega^2 + \frac{4mkc(a + c)}{I + m(a + c)^2} - k\right]\xi = 0.$$

124. A solid right circular cone of weight W, slant height l, and vertex angle 2α rolls, without slipping, on a rough inclined plane as shown in Fig. 179. Taking the angle ϕ shown in the figure as generalized coordinate, show that oscillations of the cone about its position of equilibrium are defined by the equation

$$\frac{3}{10}\cos^2\alpha\frac{l}{g}\left(3\cos^2\alpha + \frac{1}{2}\sin^2\alpha\right)\ddot{\phi} + \frac{3}{4}(\cos^2\alpha\sin\beta)\sin\phi = 0,$$

which, for small values of ϕ, represents a simple harmonic motion.

29. Constraints Depending on Time.—In discussing various kinds of constraints in Art. 23, we already mentioned that equations of constraint may contain time t explicitly. This means that instead of Eqs. (*b*) of the preceding article, we may have

$$\left.\begin{aligned} x_i &= F_i(t,q_1,q_2, \ldots q_n), \\ y_i &= G_i(t,q_1,q_2, \ldots q_n), \\ z_i &= H_i(t,q_1,q_2, \ldots q_n). \end{aligned}\right\} \qquad (a)$$

D'Alembert's principle still holds in this case, and we can say that the active forces on a system together with inertia forces and reactive forces of constraints represent, at each instant, a system of forces in equilibrium. In applying the principle of virtual displacements to such systems at a *certain instant*, we must consider as virtual displacements such small displacements that are compatible with the equations of constraint *at that instant.* This means that in calculating displacements δx_i, δy_i, δz_i,

we have to proceed as indicated by Eqs. (*c*) of the preceding article and, for each instant, consider t as a constant.

In the case of constraints independent of time t, the actual displacements that the particles of the system undergo during a small interval of time dt are compatible with constraints and represent a possible or virtual set of displacements. But this is not the case with constraints that depend on time. Virtual displacements at a certain instant t are displacements that are compatible with the conditions of constraint at *that instant*, whereas actual displacements involve variations in the constraints themselves and may not satisfy the requirements of virtual displacements at the instant t.

Take, for example, a simple pendulum the suspension point of which performs a prescribed motion (Fig. 180). A virtual displacement of the particle A, at any instant t, is a small displacement along the circle mn, the center of which coincides with the instantaneous position of the point of suspension O. The reactive force is normal to this circle, and its work on the virtual displacement vanishes. The actual displacement AA_1 during an interval dt consists of two components, one along the circle and the other together with the moving circle. We see that the actual displacement does not satisfy the requirement imposed upon virtual displacements; *i.e.*, it is not compatible with constraints at the instant t. We see also that the work of the reactive force on the actual displacement does not vanish.

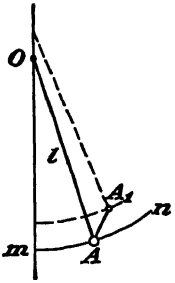


FIG. 180.

With these general remarks we see that in the case of constraints depending on time, reactions may produce work on actual displacements but not on virtual displacements and Eq. (*d*) of the preceding article holds also in this case. We have to consider only the transformation of the left-hand side of that equation to generalized coordinates. Using Eqs. (*a*) above, we obtain

$$\left.\begin{aligned}\dot{x} = \frac{dx}{dt} &= \frac{\partial x}{\partial q_1}\dot{q}_1 + \frac{\partial x}{\partial q_2}\dot{q}_2 + \cdots + \frac{\partial x}{\partial q_n}\dot{q}_n + \frac{\partial x}{\partial t},\\ \frac{\partial \dot{x}}{\partial q_s} &= \frac{\partial^2 x}{\partial q_1 \partial q_s}\dot{q}_1 + \frac{\partial^2 x}{\partial q_2 \partial q_s}\dot{q}_2 + \cdots + \frac{\partial^2 x}{\partial q_n \partial q_s}\dot{q}_n + \frac{\partial^2 x}{\partial t\, \partial q_s},\\ \frac{d}{dt}\frac{\partial x}{\partial q_s} &= \frac{\partial^2 x}{\partial q_1 \partial q_s}\dot{q}_1 + \frac{\partial^2 x}{\partial q_2 \partial q_s}\dot{q}_2 + \cdots + \frac{\partial^2 x}{\partial q_n \partial q_s}\dot{q}_n + \frac{\partial^2 x}{\partial q_s \partial t}.\end{aligned}\right\} \quad (b)$$

Comparing these equations with Eqs. (*g*), (*h*), (*i*) of the preceding article, we find that the difference consists only in the last terms on the right-hand side of our present equations, which appear due to presence of time t in Eqs. (*a*). The presence of these terms does not affect our previous

conclusions, and we have, as before

$$\frac{\partial \dot{x}}{\partial \dot{q}_s} = \frac{\partial x}{\partial q_s}, \qquad \frac{\partial \dot{x}}{\partial q_s} = \frac{d}{dt}\frac{\partial x}{\partial q_s}. \tag{c}$$

This means that Eqs. (*l*) and (*m*) of the preceding article hold also in this case and we obtain the same Lagrangian equations (135).

As an example of constraint containing time explicitly, let us consider a pendulum consisting of a particle B on a string the free length of which can be varied by pulling the end A (Fig. 181). The position of the particle is defined by the length l and the angle ϕ. Since l is a given function of time, depending on the motion of point A, we have a system with one degree of freedom and can take ϕ as the generalized coordinate. The rectangular coordinates of the particle B then are

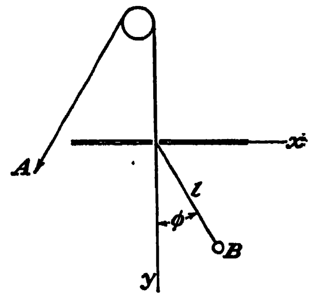


FIG. 181.

$$x = l \sin \phi, \qquad y = l \cos \phi.$$

These equations contain time explicitly, since l is a given function of time. The expression for kinetic energy of the system is

$$T = \tfrac{1}{2}m(\dot{x}^2 + \dot{y}^2) = \tfrac{1}{2}m[(\dot{l} \sin \phi + l \cos \phi \dot{\phi})^2 + (\dot{l} \cos \phi - l \sin \phi \dot{\phi})^2]$$
$$= \tfrac{1}{2}m(\dot{l}^2 + l^2\dot{\phi}^2).$$

and the potential energy is

$$V = -mg\, l \cos \phi.$$

Substituting in Eq. (136), we obtain[1]

$$\frac{d}{dt}(ml^2\dot{\phi}) = -mgl \sin \phi,$$

which gives

$$l^2\ddot{\phi} + 2l\dot{l}\dot{\phi} + gl \sin \phi = 0.$$

For small angles ϕ, we can take $\sin \phi \approx \phi$ and write

$$\ddot{\phi} + \frac{2\dot{l}}{l}\dot{\phi} + \frac{g}{l}\phi = 0. \tag{d}$$

Comparing this result with the equation for damped vibration (see page 34), we conclude that the second term containing the derivative $\dot{l}$ takes the place of the term representing damping. The magnitude of equivalent damping can be varied by selecting for l various functions of time.

[1] This equation can also be written by using the principles of angular momentum.

When l is increasing, $\dot{l}$ is positive and the second term of Eq. (d) represents true damping. In such case, the amplitude of oscillations will decrease with time. When l is decreasing, $\dot{l}$ becomes negative and we have negative damping. In this case, the amplitude of oscillations will increase with time. By an appropriate motion of point A of the string (Fig. 181), a progressive accumulation of energy in the system can be produced and the amplitude of oscillation of the pendulum will increase with time. Such an accumulation of energy results from the work done by the tensile force in the string during variation in the length l of the pendulum. Various functions of time for the length l can be imagined which will produce an accumulation of energy of the vibrating system. As an example, consider the case represented in Fig. 182 in which the angular velocity $\dot{\phi}$ of the pendulum and the velocity $\dot{l}$ of variation in its length are represented as functions of time. The period of variation of the pendulum's length is taken half that of its vibration period, and the $\dot{l}$ line is placed in such a manner with respect to the ϕ-line that the maximum negative damping effect coincides with the maximum speed of the particle B. This means that the length l is decreasing while the velocity of the particle is large and increasing while the velocity is comparatively small. Observing that the tensile force, applied at A, is working against the radial component of the weight of the particle together with its centrifugal force, it is easy to see that in the case represented in Fig. 182, the work done by the tensile force during decrease in the length l will be larger than that returned during increase in the length l. The surplus of work results in increasing the energy of the vibrating system.

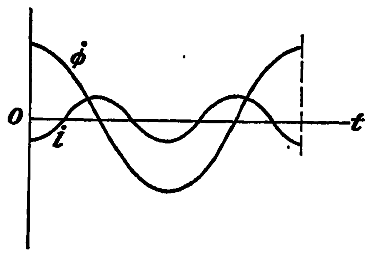


FIG. 182.

In the particular case where l is increasing or decreasing at a constant rate as for the cage of an elevator, we can put

$$l = a + bt,$$

and Eq. (d) becomes

$$\ddot{\phi} + \frac{2b}{a + bt}\dot{\phi} + \frac{g}{a + bt}\phi = 0. \tag{e}$$

This equation can be integrated by the use of Bessel's functions.

PROBLEMS

125. Show that the amplitude of vibration of the pendulum in Fig. 181 will continuously increase if each time that it passes the middle position ($\phi = 0$), its length is suddenly decreased by a small amount δ and then increased by the same amount when the pendulum is in its extreme position ($\phi = \alpha$). Show that the increase in energy per cycle is $6W\delta(1 - \cos\alpha)$.

126. Set up the equation of motion for torsional oscillations of a flywheel attached to a vertical shaft (Fig. 183) if the moment of inertia of the wheel varies periodically with time due to the harmonic motion of symmetrically located masses m sliding along two spokes of the wheel. Assume that the variable moment of inertia of the flywheel is

$$I = I_0 + 2mr^2 = I_0(1 + a \sin \omega t).$$

The torsional spring constant for the shaft is k_t; its angle of twist is θ.

Ans. $$I_0\ddot{\theta} + \frac{I_0 a\omega \cos \omega t}{1 + a \sin \omega t}\dot{\theta} + \frac{k_t}{1 + a \sin \omega t}\theta = 0.$$

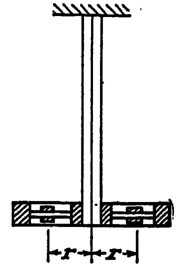


FIG. 183.

30. Lagrangian Equations for Impulsive Forces.—

Lagrangian equations (135) can easily be adapted to the case in which impulsive forces are acting on a system as discussed previously in Art. 16. For this purpose, we multiply the equations by dt and integrate them from t to $t + \epsilon$ where ϵ is a small interval of time, representing the duration of impact. In this way, for any generalized coordinate q_s, we obtain the equation

$$\int_t^{t+\epsilon} d\left(\frac{\partial T}{\partial \dot{q}_s}\right) - \int_t^{t+\epsilon} \frac{\partial T}{\partial q_s} dt = \int_t^{t+\epsilon} Q_s dt. \tag{a}$$

The first term on the left-hand side of this equation can be integrated and gives

$$\left(\frac{\partial T}{\partial \dot{q}_s}\right)_{t+\epsilon} - \left(\frac{\partial T}{\partial \dot{q}_s}\right)_t = \Delta\left(\frac{\partial T}{\partial \dot{q}_s}\right). \tag{b}$$

The second term vanishes, since we assume that the duration of impact ϵ is infinitesimal while, at the same time, $\partial T/\partial q_s$ remains finite during impact. On the other hand, the integral on the right-hand side of Eq. (a) does not vanish, since the impulsive forces that act during an impact of infinitesimal duration become infinitely large. This integral represents the impulse of the generalized impact force Q_s; we denote it by Q_s'. In calculating this impulse, we shall consider only impact forces and neglect finite forces such as gravity forces, forces of elasticity, and so on. The changes in the coordinates of the system during impact also will be neglected.[1] Equation (a) then finally gives

$$\Delta\left(\frac{\partial T}{\partial \dot{q}_s}\right) = Q_s'. \tag{138}$$

The derivative $\partial T/\partial \dot{q}_s$ represents the generalized component of the momenta corresponding to the coordinate q_s, and Eq. (138) states that the increment of this generalized component of momenta during impact is equal to the corresponding generalized component of the impulse.

[1] These are the same simplifying assumptions as were made previously in Art. 16, p. 129.

As an example, let us consider the case of a uniform rigid bar AB of length l and mass m falling in the vertical xy-plane and striking, at A, a smooth inelastic floor as shown in Fig. 184. To find the impulse Y' exerted on the bar at A, we take, as generalized coordinates, the vertical coordinate y_c of the center of gravity C and the angle of rotation θ. Then the generalized components of the impulse Y' will be obtained in the same way as in the calculation of generalized forces (see Art. 24). In this manner, we conclude that the component corresponding to the coordinate y_c is Y'. To find the component corresponding to the coordinate θ, we consider the displacement of the end A of the bar due to an increment $\delta\theta$ of the coordinates and conclude that this component is $\frac{1}{2}Y'l\sin\theta$. The kinetic energy of the system is

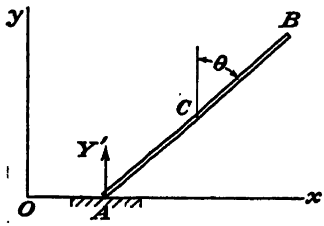


FIG. 184.

$$T = \tfrac{1}{2}m\dot{y}_c^2 + \tfrac{1}{2}I\dot{\theta}^2. \tag{c}$$

Finally, denoting the initial values of $\dot{y}_c$ and $\dot{\theta}$ just before impact by $-v$ and ω, respectively, and using expression (c) for kinetic energy together with the above generalized components of the impulse in Eq. (138), we obtain

$$\left.\begin{aligned}\Delta(I\dot{\theta}) &= I(\dot{\theta} - \omega) = \tfrac{1}{2}Y'l\sin\theta,\\ \Delta(m\dot{y}_c) &= m(\dot{y}_c + v) = Y'.\end{aligned}\right\} \tag{d}$$

Regarding the velocities $\dot{y}_c$ and $\dot{\theta}$ of the bar after impact, we know that the vertical component of the velocity at A must vanish. Hence

$$\tfrac{1}{2}\dot{\theta}l\sin\theta + \dot{y}_c = 0. \tag{e}$$

Substituting for $\dot{\theta}$ and $\dot{y}_c$ their values from Eq. (d) and observing that $I = ml^2/12$, we obtain

$$Y' = \frac{m(v - \frac{1}{2}\omega l\sin\theta)}{1 + 3\sin^2\theta}. \tag{f}$$

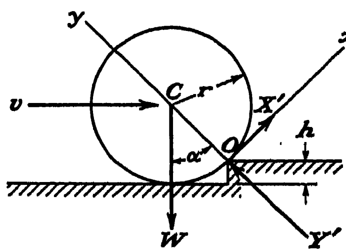


FIG. 185.

As a second example, let us consider an inelastic roller of weight W and radius r that rolls with uniform speed v along a rough horizontal plane and suddenly strikes a solid obstruction of height h as shown in Fig. 185. There ensues an impact of very short duration ϵ during which a very large reactive force, defined by components X and Y, acts at the point O as shown. During the interval of impact, the roller suffers a substantial change in its motion; and assuming sufficient friction to

prevent slipping, point O becomes the new instantaneous center of rotation. To calculate the new motion of the roller at the end of the impact and the components X' and Y' of the impulse, we can use Eqs. (138).

Taking the points of impact O as the origin of coordinate axes x and y directed as shown, we take the coordinates x_c and y_c of the center C and the angle of rotation ϕ as generalized coordinates. Then X' and Y' are the generalized forces corresponding to x_c and y_c, and the generalized force corresponding to ϕ is $X'r$. The total kinetic energy of the system at any instant is

$$T = \tfrac{1}{2}m(\dot{x}_c^2 + \dot{y}_c^2) + \tfrac{1}{2}I\dot{\phi}^2, \tag{g}$$

where m is the mass of the roller and I is its moment of inertia with respect to the geometric axis. The generalized components of momenta are

$$\frac{\partial T}{\partial \dot{x}_c} = m\dot{x}_c, \qquad \frac{\partial T}{\partial \dot{y}_c} = m\dot{y}_c, \qquad \frac{\partial T}{\partial \dot{\phi}} = I\dot{\phi}.$$

Substituting in the Lagrangian equations (138) and neglecting all finite forces during impact, we obtain (see Fig. 185)

$$\left.\begin{aligned} \Delta(m\dot{x}_c) &= m(\dot{x}_c - v\cos\alpha) = X'. \\ \Delta(m\dot{y}_c) &= m(\dot{y}_c + v\sin\alpha) = Y', \\ \Delta(I\dot{\phi}) &= \frac{mr^2}{2}\left(\dot{\phi} + \frac{v}{r}\right) = X'r. \end{aligned}\right\} \tag{h}$$

Since we assume no slipping at O during impact, these three equations are not independent; we have, as equations of constraint,

$$\dot{x}_c = -r\dot{\phi} \qquad \text{and} \qquad \dot{y}_c = 0. \tag{i}$$

Substituting in Eqs. (h), we obtain

$$\left.\begin{aligned} -m(r\dot{\phi} + v\cos\alpha) &= X', \\ mv\sin\alpha &= Y', \\ \frac{mr^2}{2}\left(\dot{\phi} + \frac{v}{r}\right) &= X'r. \end{aligned}\right\} \tag{j}$$

Substituting the value of X' from the first of these equations into the last, we obtain

$$\frac{mr^2}{2}\left(\dot{\phi} + \frac{v}{r}\right) = -m(r\dot{\phi} + v\cos\alpha)r,$$

from which

$$\dot{x}_c = -r\dot{\phi} = \frac{v}{3}(1 + 2\cos\alpha). \tag{k}$$

This equation defines the motion of the roller immediately after impact.

Using the value of $r\dot{\phi}$ from Eq. (k), in Eqs. (j), we find, for the components of the impulse at O,

$$X' = \frac{mv}{3}(1 - \cos\alpha), \qquad Y' = mv \sin\alpha. \tag{l}$$

We may now observe that to realize the assumption of no slipping at O during impact, we must have there a coefficient of friction

$$\mu \geq \frac{X'}{Y'} = \frac{1 - \cos\alpha}{3 \sin\alpha}. \tag{m}$$

If the roller were to bump a vertical wall, we would have $\alpha = 90$ deg. and the required coefficient of friction to enable it to start rolling up the wall without slipping would be $\mu \geq \frac{1}{3}$. In such case, as we see from Eq. (k), the initial velocity of the roller up the wall is only one-third of the velocity v with which it strikes the wall.

If the height h of the obstruction is small compared with the radius r of the roller, Eqs. (l) can be simplified by noting that

$$\cos\alpha = \frac{r - h}{r} = 1 - \frac{h}{r} \qquad \text{and} \qquad \sin\alpha \approx \sqrt{2}\left(\frac{h}{r}\right)^{\frac{1}{2}}.$$

Using these approximations in Eqs. (l), we obtain

$$X' = \frac{mv}{3}\left(\frac{h}{r}\right), \qquad Y' = \sqrt{2}\, mv \left(\frac{h}{r}\right)^{\frac{1}{2}}. \tag{n}$$

In such case, the coefficient of friction required to prevent slipping is

$$\mu \geq \frac{1}{3\sqrt{2}}\left(\frac{h}{r}\right)^{\frac{1}{2}} = 0.235\left(\frac{h}{r}\right)^{\frac{1}{2}}. \tag{o}$$

Having Eq. (k) defining the motion of the roller just after impact, we can easily calculate the kinetic energy that it loses as a result of the blow. To do this, we note that the kinetic energy before impact was

$$T_0 = \frac{mv^2}{2} + \frac{mr^2}{2}\frac{v^2}{2r^2} = \frac{3}{4}mv^2. \tag{p}$$

Using the values (i) and (k) in expression (g), the kinetic energy after impact is

$$T = \frac{mv^2}{12}(1 + 2\cos\alpha)^2. \tag{q}$$

Subtracting Eq. (q) from Eq. (p), we obtain, for the lost energy,

$$\Delta T = T_0 - T = \frac{mv^2}{12}[9 - (1 + 2\cos\alpha)^2]. \tag{r}$$

For small values of the ratio h/r, this expression is easily reduced to

$$\Delta T \approx mv^2\left(\frac{h}{r}\right).$$

PROBLEMS

127. A rhombus formed of four hinged bars, each of length l and mass m, falls vertically and strikes a horizontal inelastic surface as shown in Fig. 186. Find the subsequent motion if, before impact, the diagonal AB is vertical and the bars have only translatory motion with velocity $-v$. Take the coordinate y_c of the center of gravity C and the angle θ as generalized coordinates.

Ans. After impact, $\dot{\theta} = \dfrac{3v \sin \theta}{l(1 + 3 \sin^2\theta)}$.

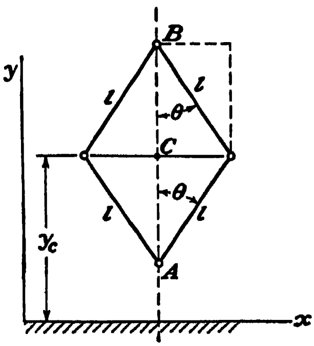


FIG. 186.

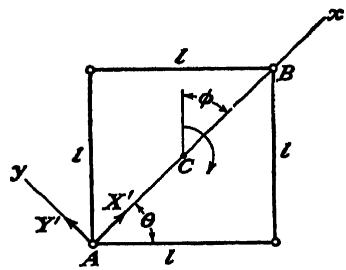


FIG. 187.

128. A square formed of four hinged bars, each of mass m and length l, rotates in its own plane with angular velocity ω about its center C as shown in Fig. 187. Find the subsequent motion if the hinge A is suddenly fixed. Generalized coordinates, defining the configuration of the system, will be the coordinates x_c and y_c of the center of the square together with the angle of rotation ϕ of its diagonal AB and the angle θ that each bar makes with this diagonal. *Ans.* After impact, $\dot{\phi} = \frac{2}{5}\omega$, $\dot{\theta} = 0$.

129. A regular hexagon, formed of six hinged bars, each of length l and mass m, is initially at rest on a perfectly smooth horizontal plane as shown in Fig. 188. Find the ratio of the velocity $\dot{y}_d$ and $\dot{y}_a$ of the mid-points of two parallel bars just after a blow of total impulse

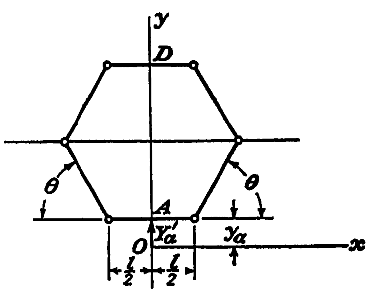


FIG. 188.

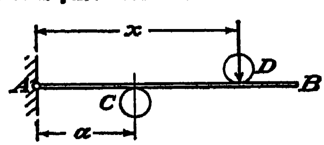


FIG. 189.

Y_a' is struck at point A. Owing to symmetry, the configuration of the system will be completely defined by the coordinate y_a of the point A at which the blow is struck together with the angle θ that each inclined bar makes with the x-axis.

Ans. $l\dot{\theta} = -\frac{9}{10}\dot{y}_a$, $\dot{y}_d = \dot{y}_a + l\dot{\theta} = \frac{1}{10}\dot{y}_a$.

130. A rigid prismatic bar AB of length l and mass m can rotate freely on a horizontal smooth surface about a vertical axis through A (Fig. 189). A ball C of mass m_c

is placed in contact with the bar at the distance a from the axis of rotation. At what distance x should another ball D of mass m_d impinge the bar in order to impart the greatest possible velocity to the ball C by the impact. Assume an inelastic impact.

$$Ans. \quad x = \sqrt{\frac{ml^2}{3m_d} + \frac{m_c a^2}{m_d}}.$$

31. Hamilton's Principle.—Multiplying each term of the Lagrangian equation (135) by an assumed virtual displacement δq_s and using notation (l) on page 213, we may write

$$\frac{d}{dt}\left[\sum m\left(\dot{x}\frac{\partial x}{\partial q_s} + \dot{y}\frac{\partial y}{\partial q_s} + \dot{z}\frac{\partial z}{\partial q_s}\right)\delta q_s\right] - \frac{\partial T}{\partial q_s}\delta q_s = Q_s\delta q_s. \tag{a}$$

If the forces have potential and V is the potential energy of the system, we have $Q_s = -\partial V/dq_s$ and Eq. (a) becomes

$$\frac{d}{dt}\left[\sum m\left(\dot{x}\frac{\partial x}{\partial q_s} + \dot{y}\frac{\partial y}{\partial q_s} + \dot{z}\frac{\partial z}{\partial q_s}\right)\delta q_s\right] = \left(\frac{\partial T}{\partial q_s} - \frac{\partial V}{\partial q_s}\right)\delta q_s. \tag{b}$$

Introducing the notation

$$L = T - V,$$

Eq. (b), in turn, may be written in the form

$$\frac{d}{dt}\left[\sum m\left(\dot{x}\frac{\partial x}{\partial q_s} + \dot{y}\frac{\partial y}{\partial q_s} + \dot{z}\frac{\partial z}{\partial q_s}\right)\delta q_s\right] = \frac{\partial L}{\delta q_s}\delta q_s. \tag{c}$$

The function L, representing the difference of kinetic and potential energies of the system, is called the *Lagrangian function*, and the right-hand side of Eq. (c) represents the increment of this function corresponding to the assumed virtual displacement δq_s.

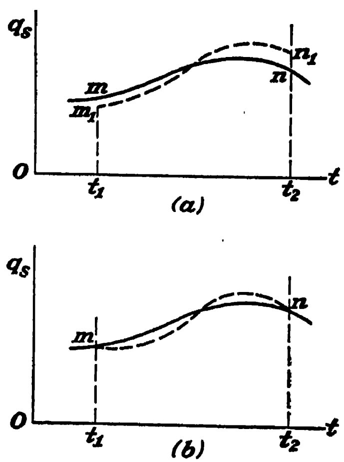


FIG. 190.

Let us consider now an interval of time between two arbitrarily chosen limits t_1 and t_2 and assume that at every instant, some virtual displacement is given to the system, so that δq_s becomes a certain function of time, the magnitude of which always remains very small. We assume also that this function does not change abruptly but is a continuous function of time and also that its derivative with respect to time is always small. Let the curve mn in Fig. 190a represent values of the chosen coordinate q_s during *actual motion* of the system for the interval of time from t_1 to t_2. Then as a result of the continuously changing virtual displacement δq_s, we shall obtain another curve very close to the curve mn. One such possible variation of the curve mn is shown in

the figure by the dotted line m_1n_1. By superposing such virtual displacements on the actual motion, not only the coordinates themselves but also their derivatives with respect to time will be slightly changed. If the chosen coordinate q_s is changed to $q_s + \delta q_s$, the corresponding velocity will be changed from $\dot{q}_s$ to $\dot{q}_s + (d/dt)\delta q_s$. As soon as we assume some function of time for δq_s, we know the change of the coordinate q_s and of the corresponding velocity $\dot{q}_s$ at every instant and can calculate the corresponding value of the expression on the right-hand side of Eq. (*c*). In our further discussion this *variation* of the Lagrangian function will be denoted by δL. Then multiplying Eq. (*c*) by dt and integrating from $t = t_1$ to $t = t_2$, we obtain

$$\left[\sum m\left(\dot{x}\frac{\partial x}{\partial q_s} + \dot{y}\frac{\partial y}{\partial q_s} + \dot{z}\frac{\partial z}{\partial q_s}\right)\delta q_s\right]_{t_1}^{t_2} = \int_{t_1}^{t_2}(\delta L)dt. \tag{d}$$

We now introduce a certain limitation for the virtual displacement δq_s and assume that it vanishes for $t = t_1$ and $t = t_2$, as illustrated by the dotted line in Fig. 190*b*. With this limitation, the left-hand side of Eq. (*d*) vanishes and we get

$$\int_{t_1}^{t_2}(\delta L)dt = 0.$$

This equation holds for any virtual displacement defined by changes δq_1, δq_2, . . . of the coordinates, provided they vanish for $t = t_1$ and $t = t_2$. We can change the order of calculation and make the integration with respect to time first and afterward evaluate the increment of that integral due to virtual displacements δq_1, δq_2, ... In this manner we obtain

$$\delta\int_{t_1}^{t_2} L\,dt = 0. \tag{139}$$

This equation represents *Hamilton's principle*. It states that the true motion of a system within a certain arbitrarily chosen interval of time is characterized by the fact that the increment of the integral $\int_{t_1}^{t_2} L\,dt$ vanishes for any continuously varying virtual displacement, provided this displacement vanishes at the limits t_1 and t_2 of the chosen interval.

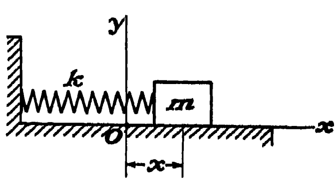


FIG. 191.

To show an application of Hamilton's principle, let us consider first a very simple example of harmonic oscillations of a mass m sliding on a smooth horizontal plane (Fig. 191). Denoting by x the displacement of the particle from its position of equilibrium, the potential and kinetic energies are

$$V = \frac{kx^2}{2} \quad \text{and} \quad T = \frac{m\dot{x}^2}{2},$$

and Hamilton's principle gives

$$\delta \int_{t_1}^{t_2} (m\dot{x}^2 - kx^2)dt = 0. \tag{e}$$

We know, of course, that in this case, the mass performs a simple harmonic motion and the coordinate x, as a function of time, is represented by a sine curve. Let us take any interval of time from t_1 to t_2 and give to x some small continuously varying increment δx such that the above-mentioned sine curve will be slightly distorted but will retain its true ordinates at $t = t_1$ and at $t = t_2$. The velocity $\dot{x}$ of the mass m will then be changed to $\dot{x} + (d/dt)(\delta x)$, and Eq. (e) gives

$$\int_{t_1}^{t_2} \left[m\left(\dot{x} + \frac{d\,\delta x}{dt}\right)^2 - k(x + \delta x)^2\right]dt - \int_{t_1}^{t_2} (m\dot{x}^2 - kx^2)dt = 0.$$

Observing that δx and $(d/dt)(\delta x)$ are small quantities and neglecting terms containing their squares, we obtain

$$\int_{t_1}^{t_2} \left(m\dot{x}\frac{d\,\delta x}{dt} - kx\,\delta x\right)dt = 0. \tag{f}$$

We conclude now that the true expression for x is that function of time which makes this integral vanish. Integration of the first term in the parentheses can be made by parts, and we obtain

$$\int_{t_1}^{t_2} m\dot{x}\frac{d\,\delta x}{dt}\,dt = m\left|\dot{x}\,\delta x\right|_{t_1}^{t_2} - \int_{t_1}^{t_2} m\ddot{x}\,\delta x\,dt.$$

Since δx vanishes for $t = t_1$ and $t = t_2$, the first part of this integration vanishes and Eq. (f) reduces to

$$\int_{t_1}^{t_2} (m\ddot{x} + kx)\delta x\,dt = 0. \tag{g}$$

This integral will vanish for every virtual displacement only if the expression in the parentheses is always equal to zero, because if it is not zero, we can assume for δx a function of time that always has the same sign as the expression in the parentheses. The expression under the integral sign will then be always positive, and the integral will not vanish. Hence to make it vanish, we must have

$$m\ddot{x} + kx = 0,$$

which is the known equation for simple harmonic motion.

Naturally, in this simple case, we do not need to use Hamilton's

principle to derive the equation of motion. In general, we shall find that the use of this principle does not simplify the solution of problems dealing with systems having a finite number of degrees of freedom. However, it may have some advantages in the case of elastic systems with an infinite number of degrees of freedom and has found some useful applications in various branches of physics.[1]

Hamilton's principle represents the most condensed form of describing the motion of a given mechanical system, and from it the usual equations of motion in rectangular coordinates and also Lagrangian equations can be derived.

Take, for example, a system of free particles under the action of forces having potential. Then

$$L = T - V = \tfrac{1}{2}\Sigma m_i(\dot{x}_i^2 + \dot{y}_i^2 + \dot{z}_i^2) - V.$$

If, during some interval of time, we give to any coordinate x_i a virtual displacement δx_i such as shown by the dotted line in Fig. 190*b*, we find, as in the above-discussed example,

$$\delta L = m_i\dot{x}_i\frac{d\,\delta x_i}{dt} - \frac{\partial V}{\partial x_i}\,\delta x_i = m_i\dot{x}_i\,\frac{d\,\delta x_i}{dt} + X_i\delta x_i$$

and, from Hamilton's principle,

$$\int_{t_1}^{t_2}\left(m_i\dot{x}_i\,\frac{d\,\delta x_i}{dt} + X_i\delta x_i\right)dt = 0.$$

Observing that

$$\int_{t_1}^{t_2} m_i\dot{x}_i\,\frac{d\,\delta x_i}{dt}dt = \left|m_i\dot{x}_i\delta x_i\right|_{t_1}^{t_2} - \int_{t_1}^{t_2} m_i\ddot{x}_i\delta x_i dt = -\int_{t_1}^{t_2} m_i\ddot{x}_i\delta x_i dt,$$

we obtain

$$\int_{t_1}^{t_2} (m_i\ddot{x}_i - X_i)\,\delta x\,dt = 0,$$

which requires that

$$m_i\ddot{x}_i - X_i = 0.$$

These are the known equations of motion in rectangular coordinates. In a similar way, Lagrangian equations also can be obtained.

We assumed in our derivation of Hamilton's principle that the forces have a potential. If, in addition to forces having potential, there are other forces that do not, we keep the symbol Q_s for these later forces and the right-hand side of Eq. (*b*) becomes

$$\left(\frac{\partial T}{\partial q_s} - \frac{\partial V}{\partial q_s} + Q_s\right)\delta q_s.$$

[1] H. HELMHOLTZ, *J. Math.*, vol. 100, 1886; *Wied. Ann.*, vol. 47, p. 1, 1892; J. J. THOMSON, "Applications of Dynamics to Physics and Chemistry," 1888.

Proceeding then as before, we shall find, for each coordinate, the equation

$$\int_{t_1}^{t_2} (\delta L + Q_s \delta q_s) dt = 0. \tag{140}$$

As an application of this equation, let us consider forced vibrations of a system with one degree of freedom and with a nonlinear spring. Proceeding as before and denoting by x the displacement from the equilibrium position (Fig. 191) and using the notations $f(x)$ and $P \sin \omega t$ for the force in the spring and for the disturbing force, respectively, we obtain

$$\int_{t_1}^{t_2} [m\ddot{x} + f(x) - P \sin \omega t] \delta x \, dt = 0 \tag{h}$$

similar to Eq. (g) above. Like Eq. (g), this requires

$$m\ddot{x} + f(x) - P \sin \omega t = 0, \tag{i}$$

which is the known differential equation for forced vibrations of a nonlinear system without damping [see Eq. (a), page 61].

Although Hamilton's principle can bring us to the usual differential equation of motion for forced vibrations of a nonlinear system, Eq. (h) furnishes us with a more powerful method of making an approximate solution of this problem.[1] Following the *Ritz method*,[2] we assume that the steady forced vibrations of the mass m can be represented with sufficient accuracy by the series

$$x = a_1 \sin \omega t + a_2 \sin 2\omega t + \cdots . \tag{j}$$

Then a virtual displacement of the mass m can be obtained by giving to any coefficient a_s in this series an increment δa_s and then we shall have

$$\delta x = \delta a_s \sin s\omega t. \tag{k}$$

Substituting expressions (j) and (k) into Eq. (h) and integrating from $t_1 = 0$ to $t_2 = 2\pi/s\omega$,* we shall obtain as many equations for calculating $a_1, a_2, \ldots$ as the number of terms in the series (j). Naturally, the more terms we take in this series the more accurately can it be made to represent the true motion. Practically, however, good results can often be obtained by taking only one or two terms.

If we take only the first term of the series (j), we have

$$x = a \sin \omega t \tag{l}$$

and

$$\delta x = \delta a \sin \omega t. \tag{m}$$

[1] We have already seen in Art. 8, that an exact solution of Eq. (i) is not to be had

[2] See WALTER RITZ, "Gesammelte Werke," 1911.

* The virtual displacement δx vanishes for these limits.

Substituting these expressions into Eq. (h) and taking the interval $t_2 - t_1 = 2\pi/\omega$, we obtain

$$\int_0^{\frac{2\pi}{\omega}} [ma\omega^2 \sin \omega t - f(x) + P \sin \omega t]\delta a \sin \omega t \, dt = 0. \tag{n}$$

Canceling δa and making part of the indicated integration, we obtain

$$ma\omega^2 + P = \frac{\omega}{\pi} \int_0^{\frac{2\pi}{\omega}} f(a \sin \omega t)\sin \omega t \, dt. \tag{141}$$

This equation can be used to determine the amplitude a of the assumed motion.

Let us apply Eq. (141) first to the steady forced vibrations of the mass particle m of the system in Fig. 44 (see page 61), in which case the spring force

$$f(x) = k_1 x + k_2 x^3$$

and, accordingly,

$$f(a \sin \omega t) = k_1 a \sin \omega t + k_2 a^3 \sin^3 \omega t.$$

Substituting in Eq. (141), and completing the integration on the right-hand side, we get

$$ma\omega^2 + P = k_1 a + \tfrac{3}{4} k_2 a^3.$$

Dividing through by m and rearranging terms, we have

$$\frac{k_1}{m} a + \frac{3}{4} \frac{k_2}{m} a^3 = \frac{P}{m} + a\omega^2. \tag{o}$$

As already discussed in Art. 8 (see page 63), this equation can be solved graphically, and we can construct a response curve (Fig. 46) showing how the amplitude of forced vibration varies with the angular frequency ω of the disturbing force.

It will be interesting to compare Eq. (o) with Eq. (40) on page 63. Following the notations of Art. 8, we see that these equations differ only in regard to the term containing a^3. We now have $\frac{3}{4}\beta$ in place of β in Eq. (40). It will be remembered that Eq. (40) was obtained by substituting an assumed solution, $x = a \cos \omega t$, directly into the differential equation of motion (37). In this way, we satisfied the equation only for values of t corresponding to extreme positions of the vibrating mass and, in the absence of damping, also for the middle position. Now, in using Eq. (o), obtained on the basis of Eq. (141), we are selecting the amplitude a of an assumed harmonic vibration in such a way as to satisfy most closely the differential equation of motion for all values of t throughout the cycle. In

short, Eq. (o) defines the best possible value of a within the limits of the assumed form of the motion. Taking two terms of the series (j) and proceeding in a similar manner, we can, of course, get a still better approximation of the motion.

A case of particular practical interest[1] is shown in Fig. 192, where the function $f(x)$ is defined by the straight lines having the slopes k_1 and k_2. Equation (141) applies to this system also; but owing to the discontinuous nature of $f(x)$ (see Fig. 192b), it will be necessary to divide the integration on the right-hand side into several parts. Analytically, the function $f(x)$ is defined as follows:

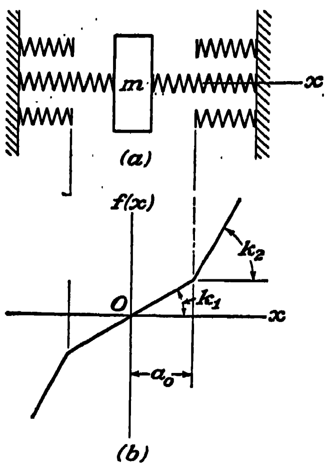


FIG. 192.

$$f(x) = \begin{cases} k_1x & \text{for} \quad -a_0 < x < a_0, \\ k_2x + (k_1 - k_2)a_0 & \text{for} \quad a_0 < x < \infty, \\ k_2x - (k_1 - k_2)a_0 & \text{for} \quad -\infty < x < -a_0. \end{cases} \tag{p}$$

Assume, now, that the amplitude a of forced vibration is greater than a_0, and let t_1 represent the time required for the mass m to move from its middle position to the point where the extra springs come into action. Setting $x = a_0$ in Eq. (l), we find that this time is the smallest value of the expression

$$t_1 = \frac{1}{\omega}\sin^{-1}\left(\frac{a_0}{a}\right).$$

Then, corresponding to expressions (p) above, we have

[1] This example is taken from a paper by A. Lourié and A. Tchekmarev, On the Approximate Solution of Several Non-linear Problems of Forced Vibrations, *Russ. Acad. Sci. Pub., Sec. Tech. Mech.*, New Series, vol. 1, No. 3, p. 307, Moscow, 1938. For another method of solution, see "Forced Vibrations in Non-linear Systems with Various Combinations of Linear Springs" by J. P. Den Hartog and R. M. Heiles, *J. Applied Mechanics*, December, 1936, p. A-127.

$$f(a \sin \omega t) = \begin{cases} k_1 a \sin \omega t & \text{for} \quad 0 < t < t_1, \\ k_2 a \sin \omega t + (k_1 - k_2)a_0 & \text{for} \quad t_1 < t < \dfrac{\pi}{\omega} - t_1, \\ k_1 a \sin \omega t & \text{for} \quad \dfrac{\pi}{\omega} - t_1 < t < \dfrac{\pi}{\omega} + t_1, \\ k_2 a \sin \omega t - (k_1 - k_2)a_0 & \text{for} \quad \dfrac{\pi}{\omega} + t_1 < t < \dfrac{2\pi}{\omega} - t_1 \\ k_1 a \sin \omega t & \text{for} \quad \dfrac{2\pi}{\omega} - t_1 < t < \dfrac{2\pi}{\omega}. \end{cases}$$

Using these expressions and dividing the integral on the right-hand side of Eq. (141) into five parts, we obtain

$$\frac{\omega}{\pi}\int_0^{\frac{2\pi}{\omega}} f(a \sin \omega t)\sin \omega t\, dt = k_2 a \pm \frac{2}{\pi}(k_1 - k_2)\left[a \sin^{-1}\left(\frac{a_0}{a}\right) + a_0\sqrt{1 - \left(\frac{a_0}{a}\right)^2}\right], \quad (q)$$

where the plus sign corresponds to positive values of a, *i.e.*, to *in-phase motion*, and the minus sign, to negative values of a, *i.e.*, to *out-of-phase motion*. Substituting in Eq. (141), we obtain, after some algebraic manipulation,

$$\frac{k_2}{k_1 - k_2}\left(\frac{\omega^2}{\omega_2^2} - 1\right)\frac{a}{a_0} + \frac{k_2}{k_1 - k_2}\left(\frac{P}{k_2 a_0}\right) = \pm\frac{2}{\pi}\left[\frac{a}{a_0}\sin^{-1}\left(\frac{a_0}{a}\right) + \sqrt{1 - \left(\frac{a_0}{a}\right)^2}\right], \quad (r)$$

where $\omega_2 = \sqrt{k_2/m}$ represents the natural angular frequency of the system for very large amplitudes where the influence of k_1 is negligible.

For amplitudes less than a_0, we have, simply,

$$f(a \sin \omega t) = k_1 a \sin \omega t,$$

and Eq. (141) gives

$$a = \frac{P}{k_1 - m\omega^2} = \frac{P/m}{\omega_1^2 - \omega^2} = \frac{P}{k_1}\left[\frac{1}{1 - (\omega^2/\omega_1^2)}\right], \quad (s)$$

where $\omega_1 = \sqrt{k_1/m}$, represents the natural angular frequency of the system within the range $-a_0 < x < a_0$. Equations (r) and (s) together define the amplitude of forced vibration in the steady state for all possible values of the angular frequency ω of the given disturbing force.

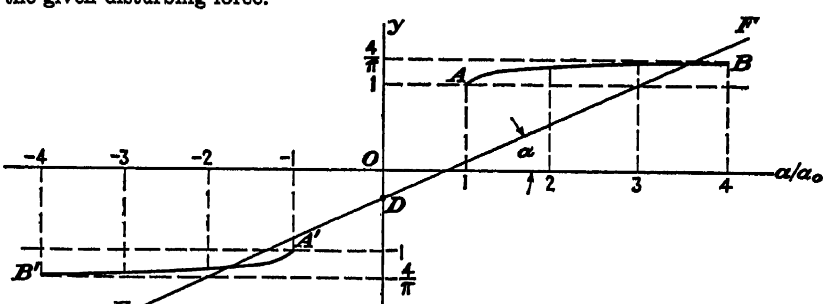


FIG. 193.

Equation (r), like Eq. (o), can be solved graphically. For this purpose, we write

$$y_1 = \pm\frac{2}{\pi}\left[\frac{a}{a_0}\sin^{-1}\left(\frac{a_0}{a}\right) + \sqrt{1 - \left(\frac{a_0}{a}\right)^2}\right],$$

which is represented graphically by the two curves AB and $A'B'$ shown in Fig. 193, where values of the amplitude ratio a/a_0 are taken as abscissa. We see that these

curves are entirely independent of the physical properties of the system and, once constructed, can be used for all cases. All physical properties of the system and the disturbing force are contained in the left-hand side of Eq. (r). We now consider the equation

$$y_2 = \frac{k_2}{k_1 - k_2}\left(\frac{\omega^2}{\omega_2^2} - 1\right)\frac{a}{a_0} + \frac{k_2}{k_1 - k_2}\left(\frac{P}{k_2 a_0}\right),$$

which represents a straight line EF having, on the y-axis, the intercept

$$OD = \frac{k_2}{k_1 - k_2}\left(\frac{P}{k_2 a_0}\right)$$

and the slope

$$\tan\alpha = \frac{k_2}{k_1 - k_2}\left(\frac{\omega^2}{\omega_2^2} - 1\right).$$

For a given system and a given disturbing force, the point D is fixed and the slope varies with the angular frequency ω. For the case shown in Fig. 192 where $k_2 > k_1$,

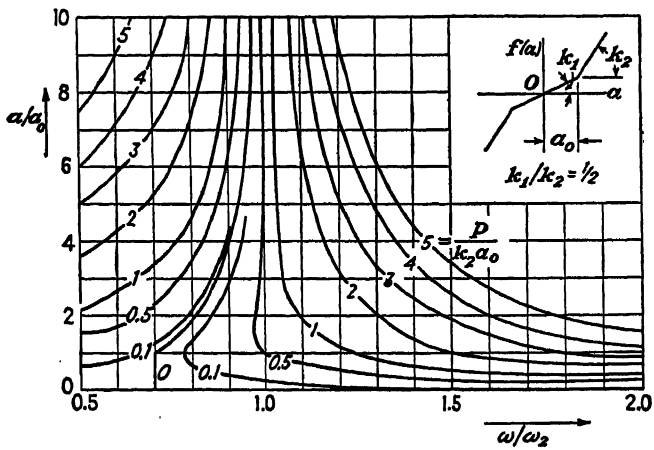


FIG. 194.

we obtain a negative intercept as shown; if $k_2 < k_1$, the intercept will be positive. The intersections between such straight lines as EF and the curves AB and $A'B'$ give all possible solutions of Eq. (r). Intersections with the branch AB indicate in-phase vibrations; those with the branch $A'B'$, out-of-phase vibrations.

Proceeding as outlined above, we can construct families of response curves showing how the amplitude of forced vibration varies with the frequency ratio ω/ω_2 for various magnitudes of the disturbing force P. One such family of curves for the case where $k_2 = 2k_1$ is shown in Fig. 194.[1] The portions of the curves below the horizontal line $a/a_0 = 1$, are obtained from Eq. (s), noting that $\omega_1/\omega_2 = 0.707$.

32. Constraints Depending on Velocities.—In discussing various kinds of constraints of a system in Art. 23, we already mentioned that equations of constraint sometimes contain not only coordinates, specifying the configuration of the system, but also velocities. Take, for example, a thin disk rolling on a rough horizontal plane as shown in Fig. 195. Any instantaneous position of this disk will be defined by the coordinates x, y of the point of contact A together with the angle ϕ that the

[1] These curves were taken from Lourié and Tchekmarev, *op. cit.*, p. 236.

tangent to the circumference of the disk at A makes with the x-axis and the angle θ by which the plane of the disk is inclined to the vertical plane. We can take these four quantities as the generalized coordinates of the system, but it may be seen that owing to friction, they will not all be independent; the system has not four but only three degrees of freedom. We cannot have arbitrary virtual displacements δx and δy, since friction opposes those displacements and the motion must take place in the direction of the tangent Om; thus

$$\delta y = \delta x \tan \phi.$$

The same relation will exist between the velocities of the point of contact A, and we have

$$\dot{y} = \dot{x} \tan \phi. \tag{a}$$

This additional equation of constraint, containing velocities, reduces the number of the degrees of freedom of the rolling disk to three.

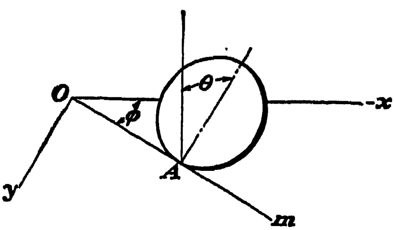


FIG. 195.

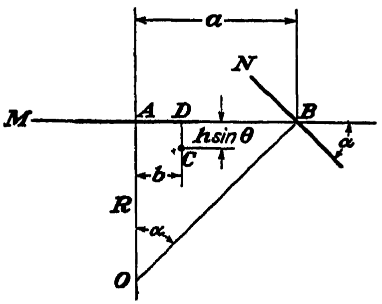


FIG. 196.

In some simple cases, the equations containing velocities can be integrated as in the example on page 193, and we obtain equations of constraint containing only coordinates as considered previously. If equations of constraint contain velocities and cannot be integrated, the problem becomes more involved.

Systems with equations of constraint containing only coordinates or coordinates and time t are called *holonomic systems*. All problems that we previously discussed in this chapter were examples of holonomic systems. The Lagrangian equations give the general method of solution of such problems. If there are equations of constraint containing velocities, such as Eq. (a), the systems are called *nonholonomic systems*. General methods of deriving equations of motion for such systems have been developed,[1] but we shall not discuss these methods here and shall limit ourselves to one example of a nonholonomic system in which an approximate solution can be obtained by using the principle of angular momentum. This is the case of motion of a bicycle (Fig. 196).[2] Let A be the point of contact of the rear wheel with the horizontal supporting plane and AM the line of intersection of the middle plane of the frame with the horizontal supporting plane. The angle of inclination of the middle plane to the vertical we denote by θ* and assume that the center of gravity C of the system is

[1] See E. J. ROUTH, Rigid Dynamics, vol. 1, p. 339, 1897; and PAUL APPELL, "Mécanique rationelle," vol. 2, p. 374, 1923.

[2] This approximate solution is given in the book "Cours de mécanique" by Bouasse, p. 620, Paris.

* The quantity θ is considered positive if inclination is toward the center of curvature O of the trajectory of point A.

in the middle plane at the distance h from the line AM, so that its height above the supporting plane is $h \cos \theta$ and its distance from the transverse vertical plane through AO is b. In an approximate discussion of the problem, we shall consider only the motion of the center of gravity of the system and shall neglect rotation of the wheels. We take the x-axis approximately parallel to the element ds of the trajectory s, which the point of contact A is describing at the instant under consideration, and the y-axis perpendicular to x. Let x and y be the coordinates of the point A. Then the cosine and sine of the angle ϕ that the element ds makes with the x- and y-axes are dx/ds, dy/ds, and the coordinates ξ, η, ζ of the center of gravity C will be

$$\left.\begin{aligned} \xi &= x + b\frac{dx}{ds} - h \sin\theta \frac{dy}{ds}, \\ \eta &= y + b\frac{dy}{ds} + h \sin\theta \frac{dx}{ds}, \\ \zeta &= h \cos\theta. \end{aligned}\right\} \tag{b}$$

To simplify writing we shall use, in further discussion, the notations

$$\frac{dx}{ds} = x', \qquad \frac{d^2x}{ds^2} = x''. \tag{c}$$

We observe also that

$$\frac{d}{dt} = \frac{d}{ds}\frac{ds}{dt} = v\frac{d}{ds}, \tag{d}$$

where v is the velocity of the point A.

To eliminate the unknown reactions, we use the principle of angular momentum and take, as an axis of moments, the line AB passing through the points of contact A and B. In such case, the moment of external forces reduces to the moment of the gravity force, equal to $Wh \sin\theta$, and the principle of angular momentum gives the following equation

$$\frac{W}{g}\frac{d^2\eta}{dt^2} h \cos\theta - \frac{W}{g}\frac{d^2\zeta}{dt^2} h \sin\theta = Wh \sin\theta.$$

Substituting for ζ its expression from Eqs. (b) and considering h as a constant, we obtain

$$\frac{d^2\eta}{dt^2}\cos\theta + h \sin^2\theta \frac{d^2\theta}{dt^2} + h \sin\theta \cos\theta \left(\frac{d\theta}{dt}\right)^2 = g \sin\theta. \tag{e}$$

To obtain an equation containing only θ, we express $d^2\eta/dt^2$ through θ by using the second of Eqs. (b), which gives

$$\frac{d\eta}{dt} = \frac{dy}{dt} + b\frac{d}{dt}\left(\frac{dy}{ds}\right) + h \sin\theta \frac{d}{dt}\left(\frac{dx}{ds}\right) + h\frac{dx}{ds}\cos\theta \frac{d\theta}{dt}$$

or, using notations (c) and (d),

$$\frac{d\eta}{dt} = vy' + bvy'' + hv \sin\theta x'' + hx' \cos\theta \frac{d\theta}{dt}.$$

The second differentiation then gives

$$\begin{aligned} \frac{d^2\eta}{dt^2} = \frac{dv}{dt}y' + v^2y'' + b\left(\frac{dv}{dt}y'' + v^2y'''\right) + h\left(\frac{dv}{dt}x'' + v^2x'''\right)\sin\theta \\ + 2hvx''\cos\theta\frac{d\theta}{dt} + hx'\left[\frac{d^2\theta}{dt^2}\cos\theta - \left(\frac{d\theta}{dt}\right)^2\sin\theta\right]. \end{aligned} \tag{f}$$

This expression can be simplified if we observe that

$$x'^2 + y'^2 = (\cos\phi)^2 + (\sin\phi)^2 = 1. \qquad (g)$$

We have also

$$x'' = -\sin\phi\,\frac{d\phi}{ds}, \qquad y'' = \cos\phi\,\frac{d\phi}{ds},$$

from which

$$(x'')^2 + (y'')^2 = \left(\frac{d\phi}{ds}\right)^2 = \frac{1}{R^2}, \qquad (h)$$

where R is the radius of curvature of the trajectory s of the point of contact A.

Differentiating Eq. (g), we obtain

$$x'x'' + y'y'' = 0, \qquad x'x''' + y'y''' + (x'')^2 + (y'')^2 = 0;$$

and by differentiating Eq. (h), we have

$$v(x''x''' + y''y''') = \frac{1}{R}\frac{d}{dt}\left(\frac{1}{R}\right). \qquad (i)$$

Assume now that the x-axis coincides with the line AB in Fig. 196. Then the angle ϕ vanishes, and we obtain

$$x' = 1, \qquad y' = 0, \qquad x'' = 0, \qquad y'' = \frac{d\phi}{ds} = \frac{1}{R}, \qquad x''' = -\left(\frac{d\phi}{ds}\right)^2$$
$$= -(y'')^2 = -\frac{1}{R^2}.$$

In such case, Eq. (i) gives

$$vy''' = \frac{d}{dt}\left(\frac{1}{R}\right)$$

and we obtain, from Eq. (f),

$$\frac{d^2\eta}{dt^2} = \frac{v^2}{R} + b\left[\frac{1}{R}\frac{dv}{dt} + v\frac{d}{dt}\left(\frac{1}{R}\right)\right] - \frac{hv^2}{R^2}\sin\theta + h\left[\frac{d^2\theta}{dt^2}\cos\theta - \left(\frac{d\theta}{dt}\right)^2\sin\theta\right]. \qquad (j)$$

Substituting this into Eq. (e), we obtain an equation containing only the variable angle θ. The equation is a complicated one, but it can be greatly simplified if we consider only small values of θ, *i.e.*, small deviations of the rear wheel from the vertical plane. In such case, we can put

$$\cos\theta = 1, \qquad \sin\theta = \theta$$

and can neglect $h\sin\theta/R$ in comparison with unity. Then

$$\frac{d^2\eta}{dt^2} = \frac{v^2}{R} + b\frac{d}{dt}\left(\frac{v}{R}\right) + h\frac{d^2\theta}{dt^2} - h\theta\left(\frac{d\theta}{dt}\right)^2,$$

and Eq. (e) gives

$$\frac{d^2\theta}{dt^2} + \frac{b}{h}\frac{d}{dt}\left(\frac{v}{R}\right) = \frac{g\theta}{h} - \frac{v^2}{hR}. \qquad (k)$$

This equation can be represented in a form more suitable for our further discussion by introducing, instead of the radius of curvature R, its expression through the angle α between the lines MB and NB. We show first that the perpendicular BO to the line NB intersects the perpendicular AO at point O, which is the center of curvature of the trajectory s of the point A. For this purpose, we denote, as before, by x and y the coordinates of point A. The coordinates of point B we denote by x_1 and y_1 and the length $\overline{AB}$ by a. Then

$$\left.\begin{aligned} x_1 &= x + a\frac{dx}{ds}, \\ y_1 &= y + a\frac{dy}{ds}, \\ (x_1 - x)^2 + (y_1 - y)^2 &= a^2. \end{aligned}\right\} \tag{l}$$

Differentiating the last of these equations and using the first two, we obtain,

$$dx(dx_1 - dx) + dy(dy_1 - dy) = 0,$$

from which

$$dx\,dx_1 + dy\,dy_1 = dx^2 + dy^2 = ds^2$$

or

$$\frac{dx}{ds}\frac{dx_1}{ds} + \frac{dy}{ds}\frac{dy_1}{ds} = 1. \tag{m}$$

Considering now the trajectory s_1 of the point of contact B of the front wheel and denoting by ds_1 an element of this trajectory, the cosines of the angles that this element makes with the x- and y-axes are

$$\frac{dx_1}{ds_1} \quad \text{and} \quad \frac{dy_1}{ds_1},$$

and we find, for the cosine of the angle α, the following expression:

$$\cos\alpha = \frac{dx}{ds}\frac{dx_1}{ds_1} + \frac{dy}{ds}\frac{dy_1}{ds_1}. \tag{n}$$

Comparing Eqs. (m) and (n), we conclude that

$$ds = ds_1 \cos\alpha. \tag{o}$$

Differentiating now the first two of Eqs. (l), squaring them, and adding together, we find

$$\begin{aligned} a^2\left[\left(\frac{d^2x}{ds^2}\right)^2 + \left(\frac{d^2y}{ds^2}\right)^2\right] &= \left(\frac{dx_1}{ds} - \frac{dx}{ds}\right)^2 + \left(\frac{dy_1}{ds} - \frac{dy}{ds}\right)^2 \\ &= \left(\frac{dx_1}{ds_1}\frac{1}{\cos\alpha} - \frac{dx}{ds}\right)^2 + \left(\frac{dy_1}{ds_1}\frac{1}{\cos\alpha} - \frac{dy}{ds}\right)^2. \end{aligned} \tag{p}$$

Assuming now that the x-axis coincides with the line AM, we obtain

$$\frac{dx_1}{ds_1} = \cos\alpha, \qquad \frac{dx}{ds} = 1, \qquad \frac{dy_1}{ds_1} = \sin\alpha, \qquad \frac{dy}{ds} = 0,$$

and Eq. (p) gives

$$a^2\left[\left(\frac{d^2x}{ds^2}\right)^2 + \left(\frac{d^2y}{ds^2}\right)^2\right] = \frac{a^2}{R^2} = \tan^2\alpha$$

or

$$a = R\tan\alpha. \tag{q}$$

The same conclusion can be obtained also from the right triangle ABO in Fig. 196, which proves that the intersection point O of the perpendiculars AO and BO is the center of curvature of the trajectory s. If we keep the angle α constant, both the trajectories s and s_1 become concentric circles of radii R and $R/\cos\alpha$, respectively.

Using Eq. (q), we rewrite now our equation of motion (k) in the following form:

$$\frac{d^2\theta}{dt^2} + \frac{b}{h}\frac{d}{dt}\left(\frac{v\tan\alpha}{a}\right) = \frac{g\theta}{h} - \frac{v^2}{hR}. \tag{r}$$

We obtain a very simple solution of this equation by assuming θ, α, and v constant. The bicycle is moving with constant velocity, with constant angle of inclination of the plane of the frame to the vertical, and with a constant angle α. Equation (r) is then satisfied if we put

$$\frac{g\theta}{h} - \frac{v^2}{hR} = 0. \tag{s}$$

This same conclusion may be reached in a very simple manner by equating the moments, with respect to the axis AB, of the gravity force and of the centrifugal force applied at the center of gravity C.

We can discuss also some more complicated cases if we only assume that the angle α is small and that the velocity v is approximately constant. Substituting α for $\tan \alpha$ and the average velocity v_a for v in Eq. (r), we obtain

$$\frac{d^2\theta}{dt^2} + \frac{bv_a}{ha}\frac{d\alpha}{dt} = \frac{g}{h}\left(\theta - \frac{v_a^2\alpha}{ga}\right). \tag{t}$$

It is seen that by a proper variation of the angle α, we can make the bicycle move with a constant angle of inclination θ of the frame. When θ is constant, Eq. (t) gives

$$\frac{d\alpha}{dt} = \frac{ag}{bv_a}\left(\theta - \frac{v_a^2\alpha}{ga}\right);$$

and by integration we obtain,

$$\alpha = \frac{ag\theta}{v_a^2} + Ce^{-\frac{v_a t}{b}}, \tag{u}$$

where C is a constant of integration which can be determined if the initial value of α is known. We see that the term containing C diminishes with time and the angle α approaches the value satisfying the condition of equilibrium (s) between the gravity force and the centrifugal force.

Returning now to Eq. (t), we assume that the right-hand side of this equation does not vanish; *i.e.*, there is no equilibrium of the gravity force and centrifugal force. If at the same time α is constant, the term containing $d\alpha/dt$ vanishes and we can integrate the equation. In this way, we shall obtain for θ an exponential function which indicates that the condition is unstable. If the right-hand side of the equation is positive and $d\theta/dt$ is positive, the angle θ will be increasing and the bicycle will be falling down. To prevent this, we have to change the sign of $d\theta/dt$. This can be accomplished by utilizing, in Eq. (t), the term containing $d\alpha/dt$. By rapidly increasing the angle α, we can make $d^2\theta/dt^2$ negative, which will finally change the sign of $d\theta/dt$. By rapidly turning the front wheel in the direction in which the bicycle is falling, we can stop the increase of the angle θ. We see that the necessary stability of the moving bicycle can be established by proper turning of the front wheel.

We neglected in our discussion the rotation of the wheels with respect to their axes and considered only the motion of the center of gravity of the system. This is by far the most important factor, and the approximate equation (t) gives a satisfactory explanation of the stability of a bicycle.

CHAPTER IV

THEORY OF SMALL VIBRATIONS

33. Free Vibrations of Conservative Systems.—Lagrangian equations, discussed in the preceding chapter, find a broad application in the study of vibrations of various mechanical systems. We begin with the case of free vibrations and assume that the particles of a system, which was initially in a configuration of *stable equilibrium*, are given some small initial displacements and velocities. Then the system will begin to perform small vibrations about its configuration of equilibrium. We assume also that there is no friction and no external *disturbing forces*, so that vibrations are due only to initial displacements and velocities. In this way, we obtain the case of *free vibrations without damping*. Under these assumptions, we usually have to consider only such forces as the mutual attractions between particles, elasticity forces of springs inserted between parts of the system, and gravity forces. All these forces have potential and the Lagrangian equations (136) can be written in the following form:

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_s} - \frac{\partial T}{\partial q_s} + \frac{\partial V}{\partial q_s} = 0. \tag{a}$$

In deriving the expressions for kinetic and potential energy of the system, we shall take into consideration that vibrations are assumed small and shall simplify the expressions for T and V by retaining only terms containing displacements and velocities in their lowest powers, neglecting small quantities of higher order.[1]

Beginning with the kinetic energy of a system, we have

$$T = \tfrac{1}{2}\Sigma m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2). \tag{b}$$

Introducing now generalized coordinates, we limit our consideration to those cases in which constraints do not depend explicitly on time, so that rectangular coordinates are functions of the generalized coordinates $q_1, q_2, \ldots q_n$ only. Then, for any particle of the system, we have

$$\left.\begin{aligned} x &= f_1(q_1,q_2, \ldots q_n), \\ y &= f_2(q_1,q_2, \ldots q_n), \\ z &= f_3(q_1,q_2, \ldots q_n), \end{aligned}\right\} \tag{c}$$

[1] Usually, the expressions for T and V will thereby become functions of the second degree in displacements and velocities, but this will not always be so.

and the corresponding velocity components are

$$\left.\begin{aligned}\dot{x} &= \frac{\partial x}{\partial q_1}\dot{q}_1 + \frac{\partial x}{\partial q_2}\dot{q}_2 + \cdots \frac{\partial x}{\partial q_n}\dot{q}_n,\\ \dot{y} &= \dots\dots\dots\dots\dots\dots\dots\dots,\\ \dot{z} &= \dots\dots\dots\dots\dots\dots\dots\dots,\end{aligned}\right\} \qquad (d)$$

We see that the velocities $\dot{x}$, $\dot{y}$, $\dot{z}$ are homogeneous linear functions of the generalized velocities $\dot{q}_1, \dot{q}_2, \ldots$. The differential coefficients such as $\partial x/\partial q_1$, $\partial x/\partial q_2$, ... in these expressions are not constant but are functions of $q_1, q_2, \ldots$ and are obtained from Eqs. (c). Making the sum of the squares of expressions (d) for the velocity components of each particle of the system and substituting in Eq. (b), we obtain, after summation, a homogeneous function of second degree in $\dot{q}_1, \dot{q}_2, \ldots$ for the kinetic energy. This can be written in the form

$$T = \tfrac{1}{2}(a_{11}\dot{q}_1^2 + a_{22}\dot{q}_2^2 + \cdots a_{nn}\dot{q}_n^2 + 2a_{12}\dot{q}_1\dot{q}_2 + \cdots), \qquad (142)$$

where $a_{11}, a_{22}, \ldots$ will be certain functions of the generalized coordinates $q_1, q_2, \ldots q_n$.

Let us introduce now the limitation of small vibrations and assume that the coordinates $q_1, q_2, \ldots$ are measured from the position of equilibrium about which the system vibrates. Then $q_1, q_2, \ldots q_n$ remain always small during vibration, and the coefficients $a_{11}, a_{22}, \ldots,$ with sufficient accuracy, can be treated as constants. To show this, we take a specific example and consider torsional vibrations of a circular disk attached to an elastic vertical shaft and connected to a piston B as shown in Fig. 197. This system has one degree of freedom; and as the generalized coordinate defining its configuration, we take the angle θ which the radius OA makes with its equilibrium position OA_0 as shown in the figure. Then the kinetic energy of the disk is $\frac{1}{2}I\dot{\theta}^2$. The kinetic energy of the connecting rod and of the piston, as we have seen before (page 174), depend not only on the angular velocity $\dot{\theta}$ of the disk but also on the angle $\alpha + \theta$ that OA makes with the fixed axis OB and can be expressed in the form $\frac{1}{2}\Theta\dot{\theta}^2$, where Θ is a certain function of the angle $\alpha + \theta$. The total kinetic energy of the system then is

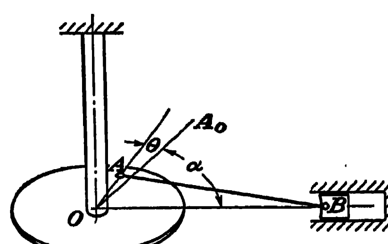


FIG. 197.

$$\tfrac{1}{2}(I + \Theta)\dot{\theta}^2. \qquad (e)$$

Now if θ is always small, there occur only small variations in the angle

$\alpha + \theta$; and with sufficient accuracy, we can take for this angle its mean value α corresponding to the equilibrium position of the system. With this value, the quantity within the parentheses of expression (e) remains constant and our complicated system can be replaced by an equivalent disk having a constant moment of inertia $I + \Theta_0$. Vibrations of such a system can be readily treated.

The above conclusion can be reached also analytically. Since α is a constant, we can consider the parentheses of expression (e) as a certain function of the small angle θ and denote it by $f(\theta)$. Then a Taylor's expansion gives

$$f(\theta) = f(0) + \theta f'(0) + \tfrac{1}{2}\theta^2 f''(0) + \cdots .$$

Observing that θ is small and neglecting in the series all terms except the first, we shall use instead of $f(\theta)$ the constant value $f(0)$, which represents the value of the parentheses of expression (e) for $\theta = 0$.

The same manner of reasoning can be applied also in the case of a system with several degrees of freedom, and we can conclude that in the case of small vibrations, the coefficients a_{11}, a_{22}, . . . in expression (142) can be considered as constants.

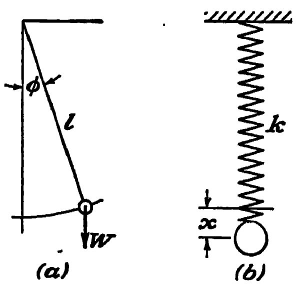


FIG. 198.

Let us consider now the expression for potential energy V. Beginning again with systems having one degree of freedom, we can readily show that in the case of small displacements measured from the position of equilibrium, the potential energy usually is a second-degree function of the coordinate. Considering, for example, a theoretical pendulum (Fig. 198a) and measuring the potential energy from the equilibrium position $\phi = 0$, we find, for any angle ϕ,

$$V = Wl(1 - \cos\phi).$$

In the case of small vibrations, the angle ϕ is small and we can take $\cos\phi \approx 1 - \frac{1}{2}\phi^2$. Then

$$V = \tfrac{1}{2}Wl\phi^2.$$

In the case of a particle on a spring (Fig. 198b), we take, for generalized coordinate, the displacement x from the position of equilibrium and the potential energy of the system is

$$V = \tfrac{1}{2}kx^2,$$

where k is the spring constant.

In general, when we have a system with one degree of freedom and q

is the coordinate measured from the position of equilibrium, we have

$$V = f(q)$$

and a Taylor's expansion gives

$$V = f(0) + f'(0)q + \tfrac{1}{2}f''(0)q^2 + \cdots . \qquad (f)$$

The first term in this expansion vanishes if we agree to measure the potential energy of the system from the position of equilibrium, *i.e.*, if we take

$$(V)_{q=0} = f(0) = 0.$$

The second term vanishes also, since q is measured from the position of equilibrium, and for such position the derivative dV/dq must vanish (see page 201). Then, neglecting all terms containing q in powers higher than the second, we obtain

$$V = \tfrac{1}{2}f''(0)q^2$$

or, using the notation $f''(0) = c$,

$$V = \tfrac{1}{2}cq^2.$$

Since, for the position of stable equilibrium, the potential energy is a minimum, $f''(0) > 0$ and the constant c is positive.

In a similar manner, we can consider a system with several degrees of freedom and conclude that in the case of small vibrations, the potential energy usually can be represented with sufficient accuracy by a homogeneous second-degree function of the coordinates. Thus, we write

$$V = \tfrac{1}{2}(c_{11}q_1^2 + c_{22}q_2^2 + \cdots + 2c_{12}q_1q_2 + \cdots), \qquad (143)$$

where the coefficients $c_{11}, c_{22}, \ldots$ are constants. Since V has a minimum for the equilibrium position, its value (143) for the displaced position will always be positive.

Substituting expressions (142) and (143) into the Lagrangian equation (*a*) and taking q_s as $q_1, q_2, \ldots q_n$, in succession, we obtain

$$\left.\begin{array}{l} a_{11}\ddot{q}_1 + a_{12}\ddot{q}_2 + a_{13}\ddot{q}_3 + \cdots + c_{11}q_1 + c_{12}q_2 + c_{13}q_3 + \cdots = 0, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots, \\ a_{n1}\ddot{q}_1 + a_{n2}\ddot{q}_2 + a_{n3}\ddot{q}_3 + \cdots + c_{n1}q_1 + c_{n2}q_2 + c_{n3}q_3 + \cdots = 0. \end{array}\right\} \qquad (144)$$

We see that keeping in the expressions for V and T only small quantities of second order, we obtain a system of *linear* equations with constant coefficients. This *linearization* of the differential equations of motion leads to a great simplification in the treatment of small vibrations.

The solution of Eqs. (144) for various particular cases will be the subject of our further discussion. First, however, we shall indicate that there are sometimes excep-

tional cases in which the potential energy can no longer be represented with sufficient accuracy by a function of the second degree but contains the coordinates $q_1, q_2, \ldots$ in higher powers. Lagrangian equations then are not linear, and their solution becomes more involved. Some problems of this kind will be discussed at the end of the chapter. Here we shall mention only a simple example of a particle C attached to a vertical string and performing horizontal vibrations along the x-axis, Fig. 199. We assume that there is no initial tension in the string and that the tension produced by the weight of the particle is small and can be neglected. Then for any horizontal displacement x of the particle, the elongation of the string is

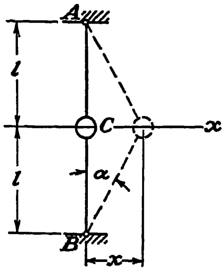


FIG. 199.

$$\lambda = 2(\sqrt{l^2 + x^2} - l) \approx \frac{x^2}{l},$$

and the tensile force will be

$$S = \frac{x^2}{2l^2} AE.$$

The corresponding potential energy of the system is

$$V = \frac{1}{2}\lambda S = \frac{x^4 AE}{4l^3}, \tag{g}$$

and the Lagrangian equation (*a*) gives

$$m\ddot{x} + \frac{AE}{l^3} x^3 = 0. \tag{h}$$

We see that in this case, the potential energy is not a second-degree function of the coordinate x and the equation of motion is not linear even for small vibrations. In our general discussion of the function representing potential energy V, we used Taylor's expansion (*f*) and concluded that for small displacements, that function usually is represented by the term $\frac{1}{2}f''(0)q^2$ and is of the second degree; but in our present example, the potential energy is given by expression (*g*). Not only the second but also the third derivative in expression (*f*) vanishes for $x = 0$, and the first term, which does not vanish and which represents V, contains the coordinate to the fourth power.

If we assume that initially the string is so highly stretched that the additional stretching due to the lateral displacement x can be neglected, the tensile force S will be approximately constant and the potential energy of the system becomes

$$V \approx \lambda S = \frac{x^2}{l} S.$$

The corresponding equation of motion is

$$m\ddot{x} + \frac{2x}{l} S = 0.$$

This equation is linear, and the particle will perform simple harmonic vibration.

From the foregoing example, we see that there can be exceptional cases in which the potential energy cannot be considered as a quadratic function of the coordinates. As mentioned above, these cases will be

discussed at the end of the chapter. We shall now consider several examples for which the kinetic and potential energies of a system can be expressed in quadratic form and to which Eqs. (144) may therefore be applied.

As a first example, consider the system in Fig. 200 where a rigid body AC suspended by a string OA performs small oscillations in a vertical plane. As generalized coordinates, we take the angles ϕ and θ as shown in the figure. Then the velocity of point A is $l\dot{\phi}$, and the velocity of the center of gravity C of the body with respect to A is $a\dot{\theta}$. The angle between these two velocities is $\theta - \phi$, and the resultant velocity v of point C is obtained from the equation

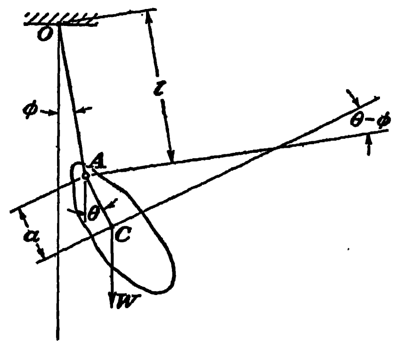


FIG. 200.

$$v^2 = l^2\dot{\phi}^2 + 2l\dot{\phi}a\dot{\theta}\cos(\theta - \phi) + a^2\dot{\theta}^2.$$

Since the middle term in this expression already contains the second-order product $\dot{\phi}\dot{\theta}$, we can take

$$\cos(\theta - \phi) \approx 1$$

and the kinetic energy of the system is

$$T = \frac{W}{2g}(l^2\dot{\phi}^2 + 2la\dot{\phi}\dot{\theta} + a^2\dot{\theta}^2) + \frac{1}{2}I\dot{\theta}^2, \tag{i}$$

where I is the moment of inertia of the body with respect to its centroidal axis. With reference to the equilibrium position, the potential energy of the system is

$$V = W[l(1 - \cos\phi) + a(1 - \cos\theta)].$$

For small oscillations, we put here

$$\cos\phi \approx 1 - \tfrac{1}{2}\phi^2, \qquad \cos\theta \approx 1 - \tfrac{1}{2}\theta^2$$

and

$$V = \tfrac{1}{2}W(l\phi^2 + a\theta^2). \tag{j}$$

We see that expressions (j) and (i) are homogeneous quadratic functions of the coordinates ϕ and θ and their derivatives $\dot{\phi}$ and $\dot{\theta}$.

Comparing expression (i) with Eq. (142), we conclude that for this case,

$$a_{11} = \frac{Wl^2}{g}, \qquad a_{22} = \left(\frac{Wa^2}{g} + I\right), \qquad a_{12} = \frac{Wal}{g}.$$

Similarly, by comparing expression (j) with Eq. (143), we find

$$c_{11} = Wl, \qquad c_{22} = Wa, \qquad c_{12} = 0,$$

and the equations of motion (144) become

$$\left.\begin{aligned} \frac{Wl^2}{g}\ddot{\phi} + \frac{Wal}{g}\ddot{\theta} + Wl\phi &= 0, \\ \left(\frac{Wa^2}{g} + I\right)\ddot{\theta} + \frac{Wal}{g}\ddot{\phi} + Wa\theta &= 0. \end{aligned}\right\} \tag{k}$$

The solution of such equations of motion will be discussed later.

As a second example, we consider a rigid prismatic bar of length $2l$ and mass m, the ends A and B of which are constrained to move along mutually perpendicular wires in a vertical plane as shown in Fig. 201. During such motion of the bar, its center of gravity C describes a circular path of radius l and with center at O. The equilibrium configuration corresponds to the lowest position of the point C on this circle; and as our generalized coordinate, we take the angle ϕ that the line OC makes with the vertical. We assume that this angle is always small. For any position of the bar, the instantaneous center of rotation is at O_1, the angular velocity is $\dot{\phi}$, and the velocity of point C is $l\dot{\phi}$. Thus the total kinetic energy is

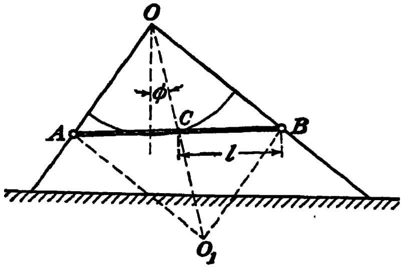


FIG. 201.

$$T = \tfrac{1}{2}ml^2\dot{\phi}^2 + \tfrac{1}{2}I\dot{\phi}^2 = \tfrac{2}{3}ml^2\dot{\phi}^2. \tag{l}$$

With reference to the equilibrium position, the potential energy is

$$V = Wl(1 - \cos\phi) \approx \frac{Wl}{2}\phi^2. \tag{m}$$

Comparing these expressions with Eqs. (142) and (143), we conclude that

$$a_{11} = \tfrac{4}{3}ml^2, \qquad c_{11} = Wl = mgl,$$

and Eq. (144) gives

$$\tfrac{4}{3}ml^2\ddot{\phi} + mgl\phi = 0,$$

which can be written in the form

$$\ddot{\phi} + \frac{3}{4}\frac{g}{l}\phi = 0. \tag{n}$$

From this equation, we conclude that for small oscillations, the system has the same period as a simple pendulum of length $4l/3$.

As a third example, we consider the case of a particle B of mass m attached to the upper end of a slender vertical wire of length l as shown in Fig. 202. The mass of the wire will be neglected, and we limit our discussion to small lateral vibrations of the particle B. Taking coordinate axes, as shown, and denoting by y_b the lateral displacement of the particle B, the kinetic energy of the system is

$$T = \tfrac{1}{2} m\dot{y}_b^2. \tag{o}$$

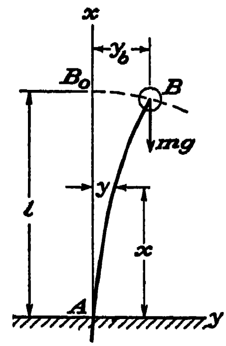


FIG. 202.

To calculate the potential energy, we have to consider, besides the gravity force mg, the strain energy of bending of the wire. This strain energy is

$$V_1 = \int_0^l \frac{M^2 dx}{2EI} = \frac{(mg)^2}{2EI} \int_0^l (y_b - y)^2 dx. \tag{p}$$

Regarding the potential energy of the gravity force mg, we observe that during vibration, the particle describes a curve B_0B as shown. The corresponding vertical displacement of the particle is

$$\lambda = \int_0^l (ds - dx) \approx \frac{1}{2} \int_0^l \left(\frac{dy}{dx}\right)^2 dx. \tag{q}$$

Assuming now, for approximate calculation, that the deflection curve has the equation

$$y = y_b\left(1 - \cos\frac{\pi x}{2l}\right),$$

Eqs. (p) and (q) give, respectively,

$$V_1 = \frac{(mg)^2 l}{4EI} y_b^2 \qquad \text{and} \qquad \lambda = \frac{\pi^2}{16l} y_b^2.$$

Then the total potential energy of the system is

$$V = \frac{1}{2}\left[\frac{(mg)^2 l}{2EI} - \frac{\pi^2 mg}{8l}\right] y_b^2. \tag{r}$$

If this expression is positive, the straight-line form of equilibrium of the column is stable and we can have lateral vibrations of the particle B with respect to that position of equilibrium. If V is negative, the straight configuration of equilibrium will be unstable and the column buckles sidewise. The critical value of the weight mg is obtained by equating

the expression in the brackets of Eq. (r) to zero. In this way, we get

$$(mg)_{cr} = \frac{\pi^2 EI}{4l^2}, \tag{s}$$

which, of course, is the known Euler's load for the column.

Using the notation

$$k = \frac{(mg)^2 l}{2EI} - \frac{\pi^2 mg}{8l}, \tag{t}$$

Eqs. (144) give

$$m\ddot{y}_b + ky_b = 0.$$

We see from this that the compressed column has an effective spring constant as given by expression (t), and the particle B vibrates with the period $\tau = 2\pi\sqrt{m/k}$. Taking mg close to the critical value (s), we obtain a very small effective spring constant and the period approaches infinity.

As a final example, we consider a prismatic bar AB of length $2c$ and mass m suspended horizontally in a vertical plane by two identical strings of length l symmetrically orientated with respect to the vertical CO through the center of gravity C as shown in Fig. 203. Let α denote the angle of inclination of each string when the system is in its configuration of equilibrium, *i.e.*, when AB is horizontal. During small oscillations of the system in the plane of the figure, each string makes a small angle ϕ with its equilibrium position, and we take this angle ϕ as generalized coordinate. The corresponding angle of inclination of the bar AB we denote by ψ. Noting from the figure that O is the instantaneous center of rotation of the bar when it is in the equilibrium position, we see that

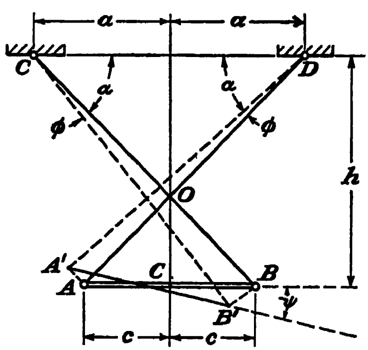


FIG. 203.

$$\overline{OC} = h\frac{c}{a+c}, \qquad \overline{OA} = \overline{OB} = l\frac{c}{a+c}.$$

Thus, the velocity of the center of gravity C of the bar is

$$h\frac{c}{a+c}\dot{\psi},$$

and the velocities of its ends A and B are

$$l\frac{c}{a+c}\dot{\psi} = l\dot{\phi},$$

so that

$$\psi = \frac{a + c}{c}\,\phi.$$

The kinetic energy of the bar then is

$$T = \frac{m}{2}\left(\frac{hc\,\dot{\psi}}{a + c}\right)^2 + \frac{I\dot{\psi}^2}{2} = \frac{m}{2}\left[h^2 + \frac{1}{3}(a + c)^2\right]\dot{\phi}^2. \qquad (u)$$

This expression, derived for the middle position $\phi = 0$, can be used with satisfactory accuracy also for small values of ϕ.

The potential energy of the system with respect to the equilibrium position is

$$V = -mg[l \sin(\alpha + \phi) - c \sin \psi - h]. \qquad (v)$$

The relation between ϕ and ψ can be found by making the projections of the closed polygon $A'B'CDA'$ on the horizontal and vertical axes. Considering then ψ and ϕ as small quantities and expanding the trigonometric functions of expression (v) in series, we shall find

$$V = mgh\frac{\phi^4}{4 \times 3 \times 2 \times 1}\,\frac{3(a + c)}{c}\,\frac{3c - a}{c} + \cdots. \qquad (w)$$

We see that the first term in this series contains the coordinate ϕ to the fourth power. The pendulum will oscillate, but not the oscillations will be harmonic. We see also that if $3c = a$, the term containing ϕ^4 in expression (w) vanishes and the first term, different from zero in this case, is that containing ϕ to the sixth power. This indicates that the trajectory of the center of gravity C is very close to a horizontal straight line and the period of oscillation will be very large.

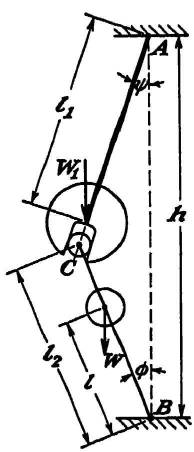


FIG. 204.

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131. For small oscillations about its vertical configuration of equilibrium, write the expressions for kinetic and potential energy of the system shown in Fig. 204; and using same, formulate the equation of motion using the angle ψ as generalized coordinate. Neglect the masses of the bars and of the roller C, and assume that there is no friction.

Ans. $\left[\frac{W_1}{g} l_1^2 + \frac{W}{g} l^2 \left(\frac{h - l_2}{l_2}\right)^2\right]\ddot{\psi} + \left[W_1 l_1 - Wl \left(\frac{h - l_2}{l_2}\right)^2\right]\psi = 0.$

132. Formulate the equation of motion for small rotational oscillations of the system in Fig. 205 about the vertical axis of symmetry through C and find the period τ. *Ans.* $\tau = 2\pi\sqrt{l/3g}$.

133. Find the period τ of small oscillations of the particle C shown in Fig. 206. Neglect friction and the mass of the bar OC. *Ans.* $\tau = 2\pi\sqrt{l/g \sin \alpha}$

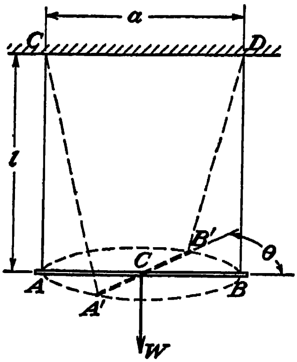

FIG. 205.

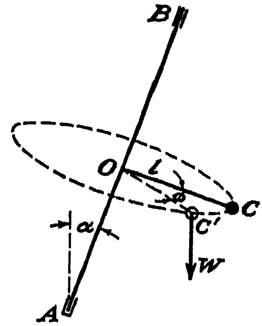

FIG. 206.

34. Linear Oscillations of Two Coupled Masses.—Before entering into a general discussion of the solution of Eqs. (144), we shall consider the particular case of small vibrations of the system shown in Fig. 207. Two masses m_1 and m_2 can slide without friction along the horizontal x-axis, and the connecting springs have the spring constants k_1 and k_2. We take as coordinates the displacements x_1 and x_2 of the masses from their positions of equilibrium, in which positions the forces in the springs vanish. For any other positions as shown in the figure, the tensile forces in the springs are k_1x_1 and $k_2(x_2 - x_1)$. The expressions for the kinetic and potential energy of the system are

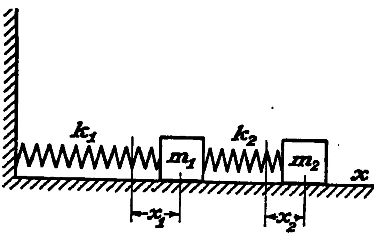

FIG. 207.

$$\left.\begin{aligned} T &= \tfrac{1}{2}(m_1\dot{x}_1^2 + m_2\dot{x}_2^2), \\ V &= \tfrac{1}{2}[k_1x_1^2 + k_2(x_2 - x_1)^2]. \end{aligned}\right\} \qquad (a)$$

Substituting in the Lagrangian equation (136), we obtain[1]

$$\left.\begin{aligned} m_1\ddot{x}_1 &= -k_1x_1 + k_2(x_2 - x_1), \\ m_2\ddot{x}_2 &= -k_2(x_2 - x_1). \end{aligned}\right\} \qquad (b)$$

To simplify our further discussion, we introduce now the notations

[1] In this simple case, the differential equations of motion can be written at once, but we use Lagrangian equations in order to make a better comparison with the general case of a system with two degrees of freedom, which will be discussed in the next article.

$$\frac{k_1 + k_2}{m_1} = a, \qquad \frac{k_2}{m_1} = b, \qquad \frac{k_2}{m_2} = c. \tag{c}$$

Then Eqs. (b) become

$$\left.\begin{aligned} \ddot{x}_1 &= -ax_1 + bx_2, \\ \ddot{x}_2 &= cx_1 - cx_2. \end{aligned}\right\} \tag{d}$$

Since we are dealing here with vibrations, it is natural to try, as particular solutions of Eqs. (d), trigonometric expressions of the form

$$\left.\begin{aligned} x_1 &= A \sin(pt + \alpha), \\ x_2 &= B \sin(pt + \alpha), \end{aligned}\right\} \tag{e}$$

which represent harmonic motions of angular frequency p, phase angle α, and amplitudes A and B. Substituting expressions (e) into Eqs. (d) and canceling $\sin(pt + \alpha)$, we obtain

$$\begin{aligned} -Ap^2 &= -Aa + Bb, \\ -Bp^2 &= Ac - Bc \end{aligned}$$

or

$$\left.\begin{aligned} A(a - p^2) - Bb &= 0, \\ -Ac + B(c - p^2) &= 0. \end{aligned}\right\} \tag{f}$$

One evident solution of these equations is $A = B = 0$. In this case, the displacements (e) vanish and we obtain the equilibrium configuration of the system. Equations (f) can yield, for A and B, solutions different from zero only if their determinant vanishes, which gives

$$(a - p^2)(c - p^2) - bc = 0 \tag{g}$$

or

$$p^4 - (a + c)p^2 + ac - bc = 0. \tag{h}$$

This equation gives for p^2 two solutions:

$$\left.\begin{aligned} p_1^2 &= \frac{a + c}{2} - \sqrt{\frac{(a + c)^2}{4} - c(a - b)}, \\ p_2^2 &= \frac{a + c}{2} + \sqrt{\frac{(a + c)^2}{4} - c(a - b)}. \end{aligned}\right\} \tag{i}$$

The expression under the radical can be transformed as follows:

$$\frac{(a + c)^2}{4} - c(a - b) = \frac{(a - c)^2}{4} + bc, \tag{j}$$

and we see that this expression is always positive; hence the roots p_1^2 and p_2^2 are always real. From notations (c), we see also that $a - b$ is always positive, which indicates that the radical in expressions (i) is smaller than $(a + c)/2$. Hence both roots p_1^2 and p_2^2 are positive, and

we get for p_1 and p_2 real values with $\pm$ signs. We need to take only the positive signs for these values to obtain the general solution of Eqs. (d) as will be discussed later.

Returning now to Eqs. (f), we see that from them we cannot obtain the amplitudes A and B but only their ratio, namely: we find

$$\frac{A}{B} = \frac{b}{a - p^2} \quad \text{or} \quad \frac{A}{B} = \frac{c - p^2}{c}.$$

For $p^2 = p_1^2$ or $p^2 = p_2^2$, both these values are equal by virtue of Eq. (g). Substituting for p^2 the values (i), we obtain two different values of the amplitude ratio, namely:

$$\frac{A_1}{B_1} = \frac{b}{a - p_1^2} = \frac{c - p_1^2}{c}, \tag{k}$$

$$\frac{A_2}{B_2} = \frac{b}{a - p_2^2} = \frac{c - p_2^2}{c}. \tag{l}$$

We see that although the magnitudes of the amplitudes are indefinite, their ratio may have only two definite values depending only on the constants a,b,c defining the physical properties of the system [see Eqs. (c)]. Taking arbitrarily two values A_1 and A_2, we obtain

$$B_1 = \frac{a - p_1^2}{b} A_1,$$

$$B_2 = \frac{a - p_2^2}{b} A_2.$$

Regarding the constant α, defining the phase of vibration, we do not get any limitation, so that finally the particular solution (e) may have either of the following two forms:

$$(x_1)' = A_1 \sin(p_1 t + \alpha_1), \qquad (x_2)' = \frac{a - p_1^2}{b} A_1 \sin(p_1 t + \alpha_1), \tag{m}$$

$$(x_1)'' = A_2 \sin(p_2 t + \alpha_2), \qquad (x_2)'' = \frac{a - p_2^2}{b} A_2 \sin(p_2 t + \alpha_2). \tag{n}$$

By virtue of Eq. (j), the radical in expressions (i) is larger than $(a - c)/2$; hence

$$a - p_1^2 = \frac{a - c}{2} + \sqrt{\frac{(a + c)^2}{4} - c(a - b)} > 0,$$

$$a - p_2^2 = \frac{a - c}{2} - \sqrt{\frac{(a + c)^2}{4} - c(a - b)} < 0,$$

and we conclude that when the system performs vibration, represented by the solution (m), the displacements of both masses m_1 and m_2 always

have the same sign while the solution (n) represents vibrations in which the displacements of the masses m_1 and m_2 always have opposite signs. It is advantageous to use a graphical representation for these two types of vibration. Taking the amplitudes of the masses m_1 and m_2 as ordinates, we obtain Figs. 208*a* and 208*b* illustrating two possible *modes* of vibration. The first one, having the smaller frequency p_1, is the *lower mode of vibration* or *fundamental mode of vibration*, and the second, with the higher frequency p_2, is the *higher mode of vibration*. These two modes of vibration are called *principal vibrations*. From expressions (m) and (n), we see that when the system performs one of the principal vibrations, the motion is harmonic. After the interval of time equal to $2\pi/p_1$ for the lower mode of vibration and to $2\pi/p_2$ for the higher mode, the system has the same configuration and the same velocities as before. The system passes twice through the configuration of equilibrium during each complete oscillation. Both particles move in the same phase; they pass simultaneously through their equilibrium positions and simultaneously reach their extreme positions; and their displacements from the equilibrium positions always are in the same ratio.

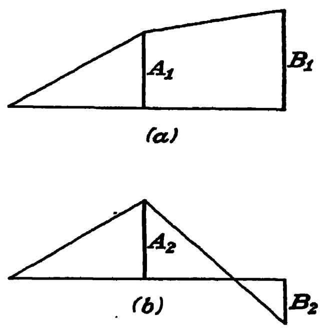

FIG. 208.

Having the two particular solutions (m) and (n), we can obtain now the general solution of Eqs. (d) by superposition, which gives

$$\left.\begin{aligned} x_1 &= A_1\sin(p_1t + \alpha_1) + A_2\sin(p_2t + \alpha_2), \\ x_2 &= \frac{a - p_1^2}{b} A_1\sin(p_1t + \alpha_1) + \frac{a - p_2^2}{b} A_2\sin(p_2t + \alpha_2). \end{aligned}\right\} \qquad (o)$$

We obtained the solution with four arbitrary constants A_1, A_2, α_1, α_2, which in each particular case must be selected so as to satisfy the initial conditions. The number of these conditions is also four, namely: we have to specify the initial displacements $(x_1)_{t=0}$, $(x_2)_{t=0}$ and the initial velocities $(\dot{x}_1)_{t=0}$, $(\dot{x}_2)_{t=0}$ of the two particles. Having these four quantities, we can, in each particular case, calculate the proper values of constants A_1, A_2, α_1, α_2, so that expressions (o) can be adapted to any initial conditions and represent the complete solution of the problem. The motion is obtained by superposition of the two principal vibrations and can be periodic only if the periods of principal vibrations are commensurable.

Naturally, we can select initial conditions such that one of the amplitudes A_1 or A_2 vanishes; then the system will perform one of the principal oscillations. If we take, for example, the initial configuration

shown in Fig. 208*a* and release the masses without initial velocities, we shall obtain $A_2 = 0$, $\alpha_1 = \pi/2$, and Eqs. (*o*) give

$$x_1 = A_1\cos p_1t, \qquad x_2 = \frac{a - p_1^2}{b} A_1\cos p_1t,$$

which indicates that the system performs, in this case, the lower mode of principal vibration.

Example: Assuming that the weights of the two particles in Fig. 207 are $W_1 = 20$ lb., $W_2 = 10$ lb. and that the spring constants are $k_1 = 200$ lb. per in. and $k_2 = 100$ lb. per in., find the principal frequencies and modes of vibration.

Solution: Substituting the given data in notations (*c*), we obtain

$$a = 15\,g \text{ in.}^{-1}, \qquad b = 5\,g \text{ in.}^{-1}, \qquad c = 10\,g \text{ in.}^{-1}$$

Equations (*i*) then give

$$p_1^2 = 5\,g \text{ in.}^{-1}, = 1{,}930 \text{ sec.}^{-2}, \qquad p_2^2 = 20\,g \text{ in.}^{-1} = 7{,}720 \text{ sec.}^{-2}$$

The number of oscillations per second will be

$$f_1 = \frac{p_1}{2\pi} = \frac{\sqrt{1{,}930}}{2\pi} \approx 7, \qquad f_2 \approx 14.$$

Substituting p_1^2 and p_2^2 into Eqs. (*k*) and (*l*), we find

$$\frac{A_1}{B_1} = \frac{1}{2}, \qquad \frac{A_2}{B_2} = -1.$$

With these values the two modes of principal vibration can be represented graphically in a manner similar to that shown in Fig. 208.

PROBLEMS

134. Calculate the frequencies and amplitude ratios for the two principal modes of torsional vibration of the system shown in Fig. 209 if $d = 5$ in., $W_1 = W_2 = 300$ lb., $l_1 = l_2 = 36$ in., and $D = 36$ in. The shafts are of steel for which the modulus of elasticity in shear is $G = 12(10)^6$ p.s.i.

Ans. $f_1 = 39.7$; $f_2 = 103.6$; $A_1/B_1 = 0.616$; $A_2/B_2 = -1.616$.

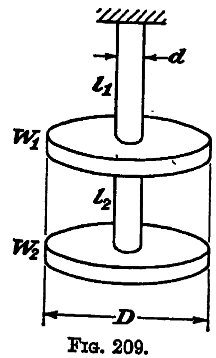

FIG. 209.

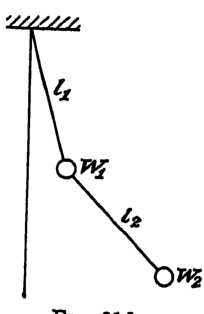

FIG. 210.

135. Calculate the frequencies and amplitude ratios for the two principal modes of oscillation of the double pendulum shown in Fig. 210 if $W_1 = W_2 = 0.386$ lb., and

$l_1 = l_2 = 10$ in. Assume that the oscillations are small and in the vertical plane of the figure. *Ans.* $f_1 \approx 0.76$; $f_2 \approx 1.83$; $A_1/B_1 = 0.707$, $A_2/B_2 = -0.707$.

35. Free Vibrations of Systems with Two Degrees of Freedom.—In the case of a system with two degrees of freedom and with constraints independent of time, expressions (143) and (142) for potential and kinetic energy become

$$\left.\begin{aligned} V &= \tfrac{1}{2}(c_{11}q_1^2 + 2c_{12}q_1q_2 + c_{22}q_2^2), \\ T &= \tfrac{1}{2}(a_{11}\dot{q}_1^2 + 2a_{12}\dot{q}_1\dot{q}_2 + a_{22}\dot{q}_2^2). \end{aligned}\right\} \tag{145}$$

For small displacements from a configuration of stable equilibrium, the coefficients of these quadratic functions usually can be considered as constants. Since we agreed to measure the potential energy from the configuration of equilibrium, for which V is a minimum, we conclude that the first of expressions (145) must be positive for any small values of q_1 and q_2. This requires that the coefficients c_{11}, c_{12}, c_{22} satisfy certain conditions. Since q_1 and q_2 are independent coordinates, we can assume $q_2 = 0$. Then, in the expression for V, there remains only the term $c_{11}q_1^2/2$; and since it must be positive, we conclude that $c_{11} > 0$. In the same manner, we can prove that $c_{22} > 0$. Assuming now that c_{22} does not vanish, we can write the first of expressions (145) in the form

$$V = \frac{1}{2c_{22}}[(c_{12}q_1 + c_{22}q_2)^2 + (c_{11}c_{22} - c_{12}^2)q_1^2],$$

from which we conclude that to have the potential energy always positive, the condition

$$c_{11}c_{22} - c_{12}^2 > 0$$

must be satisfied. Thus, altogether, we have

$$c_{11} > 0, \qquad c_{22} > 0, \qquad c_{11}c_{22} - c_{12}^2 > 0. \tag{a}$$

In a similar manner, considering the expression for kinetic energy T, which always is positive, we can prove that

$$a_{11} > 0, \qquad a_{22} > 0, \qquad a_{11}a_{22} - a_{12}^2 > 0. \tag{b}$$

Keeping conditions (a) and (b) in mind, we now return to Eqs. (144) which, for the case of a system with two degrees of freedom, become

$$\left.\begin{aligned} a_{11}\ddot{q}_1 + a_{12}\ddot{q}_2 + c_{11}q_1 + c_{12}q_2 &= 0, \\ a_{12}\ddot{q}_1 + a_{22}\ddot{q}_2 + c_{12}q_1 + c_{22}q_2 &= 0. \end{aligned}\right\} \tag{146}$$

Proceeding as in Art. 34, we assume a particular solution of these equations in the following form:

$$\left.\begin{aligned} q_1 &= A\sin(pt + \alpha), \\ q_2 &= B\sin(pt + \alpha). \end{aligned}\right\} \tag{c}$$

Then substituting into Eqs. (146), we obtain

$$\left.\begin{aligned} A(c_{11} - a_{11}p^2) + B(c_{12} - a_{12}p^2) &= 0, \\ A(c_{12} - a_{12}p^2) + B(c_{22} - a_{22}p^2) &= 0. \end{aligned}\right\} \qquad (d)$$

One evident solution of these equations is $A = B = 0$, which gives us $q_1 = q_2 = 0$, defining the configuration of equilibrium. Values of A and B different from zero can be obtained only if the determinant of Eqs. (d) vanishes, which gives

$$(c_{11} - a_{11}p^2)(c_{22} - a_{22}p^2) - (c_{12} - a_{12}p^2)^2 = 0. \qquad (147)$$

This quadratic equation in p^2 is called the *frequency equation*, and we shall now proceed to show that it has always two positive roots.

Considering the left-hand side of Eq. (147) and assuming $p^2 = 0$, we obtain $c_{11}c_{22} - c_{12}^2$, which, by the last of conditions (a), is positive. Assuming now $p^2 = +\infty$ and dividing by p^4, we obtain $a_{11}a_{22} - a_{12}^2$ which is always positive by virtue of the last of conditions (b). Thus the left-hand side of Eq. (147) is positive for $p^2 = 0$ and for $p^2 = +\infty$. On the other hand, this expression becomes negative if we take

$$p^2 = c_{11}/a_{11} \qquad \text{or} \qquad p^2 = c_{22}/a_{22}.$$

If we represent graphically the left-hand side of Eq. (147) as a function of p^2, we obtain a curve that intersects the abscissa axis twice between $p^2 = 0$ and $p^2 = \infty$. These two points define two positive roots for p^2, which we denote by p_1^2 and p_2^2. If we assume $c_{11}/a_1 > c_{22}/a_{22}$, then one of these roots, say p_1^2 will be smaller than c_{11}/a_{11} and the other will be larger than c_{22}/a_{22}.

Substituting the roots of the frequency equation (147) into either of Eqs. (d), we obtain, for the amplitude ratio A/B, the following expressions:

$$\left.\begin{aligned} \frac{A_1}{B_1} &= \frac{c_{12} - a_{12}p_1^2}{a_{11}p_1^2 - c_{11}} = \frac{c_{22} - a_{22}p_1^2}{a_{12}p_1^2 - c_{12}} = \mu_1, \\ \frac{A_2}{B_2} &= \frac{c_{12} - a_{12}p_2^2}{a_{11}p_2^2 - c_{11}} = \frac{c_{22} - a_{22}p_2^2}{a_{12}p_2^2 - c_{12}} = \mu_2. \end{aligned}\right\} \qquad (148)$$

With these ratios, the particular solution (c) can be represented in either of the following forms:

$$\left.\begin{aligned} q_1' &= A_1\sin(p_1t + \alpha_1), \\ q_2' &= \frac{a_{11}p_1^2 - c_{11}}{c_{12} - a_{12}p_1^2} A_1\sin(p_1t + \alpha_1); \end{aligned}\right\} \qquad (e)$$

$$\left.\begin{aligned} q_1'' &= A_2\sin(p_2t + \alpha_2), \\ q_2'' &= \frac{a_{11}p_2^2 - c_{11}}{c_{12} - a_{12}p_2^2} A_2\sin(p_2t + \alpha_2). \end{aligned}\right\} \qquad (f)$$

These solutions represent the two principal modes of vibration of the system. We see that for each of these modes, the two coordinates of the system are represented by harmonic functions of the same period and phase and have the amplitudes in a certain constant ratio. The magnitudes of the two principal frequencies are defined by the frequency equation (147) and are independent of the initial displacements or initial velocities given to the system.

By superposition of the principal vibrations, the general solution of Eqs. (146) is obtained. It contains four constants A_1, A_2, α_1, α_2, which can be selected in each particular case so as to satisfy the four initial conditions regarding the initial displacements $(q_1)_{t=0}$, $(q_2)_{t=0}$ and the initial velocities $(\dot{q}_1)_{t=0}$, $(\dot{q}_2)_{t=0}$.

In the preceding discussion, we assumed that the frequency equation (147) has two different roots p_1^2 and p_2^2, but there is the possibility that the two roots will be equal. This happens if

$$\frac{c_{11}}{a_{11}} = \frac{c_{22}}{a_{22}} = \frac{c_{12}}{a_{12}}.$$

Representing the left-hand side of Eq. (147) by a curve, as before, we shall find in this case that instead of two intersection points with the abscissa axis, there will be one point of tangency, the position of which defines the double root of the equation

$$p^2 = \frac{c_{11}}{a_{11}} = \frac{c_{22}}{a_{22}} = \frac{c_{12}}{a_{12}}.$$

With this value of p^2, the expressions in the parentheses of Eqs. (*d*) vanish, and these equations will be satisfied for any values of the constants A and B. We can take as a first mode of vibration the solution

$$q_1' = A_1\sin(pt + \alpha_1),$$
$$q_2' = 0$$

and for the second mode

$$q_1'' = 0,$$
$$q_2'' = A_2\sin(pt + \alpha_2).$$

The general solution in this case will be

$$q_1 = A_1\sin(pt + \alpha_1),$$
$$q_2 = A_2\sin(pt + \alpha_2).$$

The four constants A_1, A_2, α_1, α_2 have to be determined in each particular case so as to satisfy the initial conditions.

As an example of such a system, we mention the vibrations in a horizontal plane of a particle m attached to an elastic vertical bar of circular

cross section as shown in Fig. 55, page 82. As the two principal vibrations of the mass m, we can take the vibrations along the x- and y-axes. Both these vibrations have the same period, and the resulting motion of m will depend on the initial displacements $(x)_{t=0}$, $(y)_{t=0}$ and on the initial velocities $(\dot{x})_{t=0}$ and $(\dot{y})_{t=0}$.

Let us consider now another particular case for which one of the two roots of the frequency equation (147) vanishes. This occurs when the term independent of p^2 vanishes, *i.e.*, when we have

$$c_{11}c_{22} - c_{12}^2 = 0.$$

Comparing this with the last of conditions (a), we see that in this case, the configuration of the system, from which the coordinates are measured, does not represent a configuration of stable equilibrium and there is a possibility of motion without vibration.

As an example of this kind, let us consider a shaft with two disks as shown in Fig. 175, page 220. If we take, for coordinates, the angles ϕ_1 and ϕ_2 of rotation of the disks, we obtain

$$T = \tfrac{1}{2}(I_1\dot{\phi}_1^2 + I_2\dot{\phi}_2^2),$$
$$V = \tfrac{1}{2}k\,(\phi_2 - \phi_1)^2,$$

where k is the torsional spring constant of the shaft. The equations of motion are

$$I_1\ddot{\phi}_1 = k(\phi_2 - \phi_1),$$
$$I_2\ddot{\phi}_2 = -k(\phi_2 - \phi_1).$$

Substituting in these equations a particular solution of the form represented by Eqs. (c), we obtain

$$A(k - I_1p^2) - kB = 0,$$
$$-Ak + (k - I_2p^2)B = 0.$$

Then the frequency equation is

$$(k - I_1p^2)(k - I_2p^2) - k^2 = 0,$$

the roots of which are

$$p_1^2 = 0, \qquad p_2^2 = \frac{k(I_1 + I_2)}{I_1I_2}.$$

The root $p_1^2 = 0$ corresponds to the possibility for the system as a whole to rotate as a rigid body with respect to the axis of the shaft. The second root p_2^2 gives the frequency of torsional vibration of the system that can be superposed on the rigid body rotation.

The generalized coordinates q_1 and q_2, determining the configuration of a system with two degrees of freedom, can be chosen in various ways;

one particular choice is especially advantageous for analytic discussion. Imagine that instead of arbitrary coordinates q_1 and q_2, new coordinates ϕ_1 and ϕ_2 are chosen in such a manner that the terms containing products of coordinates and velocities in expressions (145) vanish. Then the expressions for potential and kinetic energy are

$$\left.\begin{aligned} V &= \tfrac{1}{2}(b_{11}\phi_1^2 + b_{22}\phi_2^2), \\ T &= \tfrac{1}{2}(d_{11}\dot{\phi}_1^2 + d_{22}\dot{\phi}_2^2), \end{aligned}\right\} \tag{g}$$

and the equations of motion are of the form

$$\left.\begin{aligned} d_{11}\ddot{\phi}_1 + b_{11}\phi_1 &= 0, \\ d_{22}\ddot{\phi}_2 + b_{22}\phi_2 &= 0. \end{aligned}\right\} \tag{h}$$

Each equation now contains only one coordinate and can be integrated independently of the other. Coordinates chosen in this manner are called *principal* or *normal coordinates.*

Having coordinates q_1 and q_2, the determination of the principal coordinates ϕ_1 and ϕ_2 is a purely algebraic problem. We put

$$\left.\begin{aligned} q_1 &= \phi_1 + \phi_2, \\ q_2 &= \beta_1\phi_1 + \beta_2\phi_2, \end{aligned}\right\} \tag{i}$$

where β_1 and β_2 are undetermined constants. Substituting these expressions into Eqs. (145), we determine the values of β_1 and β_2 so as to make the terms containing the products $\phi_1\phi_2$ and $\dot{\phi}_1\dot{\phi}_2$ vanish. With these values of β_1 and β_2 substituted into Eqs. (i), we obtain the relations between the arbitrary coordinates q_1, q_2 and the principal coordinates ϕ_1 and ϕ_2.

We can arrive at the notion of principal coordinates also in another way. From the preceding general discussion of vibrations of systems with two degrees of freedom, we know that the general case of vibration is obtained by superposition of the two principal vibrations in each of which there is a definite ratio between the coordinates q_1 and q_2 given by Eqs. (148). In the example of the preceding article, the two principal modes of vibration were illustrated by Fig. 208. Considering the lower mode of vibration, we can put

$$q_1 = \phi_1, \qquad q_2 = \frac{\phi_1}{\mu_1}. \tag{j}$$

We see that the configuration of the system at any instant is completely defined by the quantity ϕ_1, and we can take ϕ_1 as one of the coordinates. In the same manner, considering the higher mode of vibration, we can put

$$q_1 = \phi_2, \qquad q_2 = \frac{\phi_2}{\mu_2} \tag{k}$$

and can take ϕ_2 as the second coordinate. The general case of vibration is obtained by superposition of the two principal modes (j) and (k), and we get

$$q_1 = \phi_1 + \phi_2, \qquad q_2 = \frac{\phi_1}{\mu_1} + \frac{\phi_2}{\mu_2}.$$

Substituting these expressions in Eqs. (145), we shall find that the terms containing products $\phi_1\phi_2$ and $\phi_1\phi_2$ will vanish and we shall get two independent differential equations of the form (g). This is as it should be, since the two principal modes of vibration, defined by coordinates ϕ_1 and ϕ_2, are two independent harmonic motions. From this we conclude that ϕ_1 and ϕ_2 are the two principal coordinates of the system.

Take, as an example, the problem of two coupled masses (Fig. 207) already discussed in the preceding article. The coordinates x_1 and x_2, used in that discussion, are not principal coordinates, and the obtained differential equations (b) were not independent of each other. Instead of defining the configuration of the system by the displacements x_1 and x_2, we can obtain any configuration by superposing the two principal modes of vibration, illustrated in Fig. 208, and given by Eqs. (m) and (n). Considering the lower mode of vibration (m), we can put

$$(x_1)' = \phi_1, \qquad (x_2)' = \frac{a - p_1^2}{b}\,\phi_1.$$

In the same manner, for the higher mode of vibration we obtain

$$(x_1)'' = \phi_2, \qquad (x_2)'' = \frac{a - p_2^2}{b}\,\phi_2.$$

By superposition of these principal modes of vibration, we obtain

$$\left.\begin{aligned} x_1 &= \phi_1 + \phi_2, \\ x_2 &= \frac{a - p_1^2}{b}\,\phi_1 + \frac{a - p_2^2}{b}\,\phi_2. \end{aligned}\right\} \qquad (l)$$

To prove that the quantities ϕ_1 and ϕ_2 are the *principal coordinates* of the system, we substitute expressions (l) into expressions (a) of the preceding article for kinetic and potential energy of the system. A simple calculation will show that the terms containing the products of velocities and the products of coordinates are

$$\left.\begin{aligned} &m_1\left(1 + \frac{a - p_1^2}{c}\,\frac{a - p_2^2}{b}\right)\phi_1\phi_2, \\ &k_1\left[1 + \frac{b}{a-b}\left(\frac{a - p_1^2}{b} - 1\right)\left(\frac{a - p_2^2}{b} - 1\right)\right]\phi_1\phi_2. \end{aligned}\right\} \qquad (m)$$

Using expressions (*i*) of the preceding article for the two roots of the frequency equation, we obtain

$$p_2^2 = a + c - p_1^2. \tag{n}$$

Substituting this into the first of expressions (*m*), we obtain

$$\frac{m_1}{bc}\,[bc - (a - p_1^2)(c - p_1^2)]\,\phi_1\phi_2.$$

This expression vanishes by virtue of Eq. (*g*) of the preceding article. Substituting the value (*n*) into the second of expressions (*m*), we can show that this expression vanishes also. Hence, the expressions for V and T have the form of expressions (*g*), and the coordinates ϕ_1 and ϕ_2 are the principal coordinates of the system.

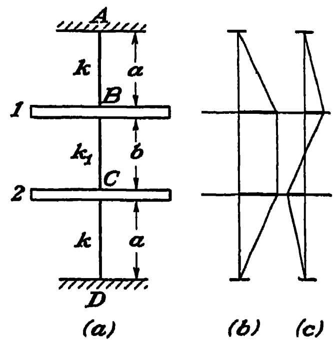

FIG. 211.

Example: A double torsional pendulum (Fig. 211) consists of two horizontal bars B and C suspended on a vertical wire AD with fixed ends. Find the periods of the two principal modes of free vibration and the general solution for the case where the bars are initially rotated in horizontal planes by the angles $(\phi_1)_0$ and $(\phi_2)_0$ from their positions of equilibrium and then released without initial velocities.

Solution: We denote by I_1 and I_2 the moments of inertia of the bars B and C with respect to the axis AD, by ϕ_1 and ϕ_2 their angles of rotation, and by k, k, and k_1 the torsional spring constants for the portions AB, CD, and BC of the wire, respectively. Then, neglecting the mass of the wire, we obtain

$$\left.\begin{aligned} T &= \tfrac{1}{2}(I_1\dot\phi_1^2 + I_2\dot\phi_2^2),\\ V &= \tfrac{1}{2}[k(\phi_1^2 + \phi_2^2) + k_1(\phi_2 - \phi_1)^2]. \end{aligned}\right\} \tag{o}$$

Comparing these expressions with Eqs. (145), we conclude that

$$\begin{aligned} a_{11} &= I_1, & a_{12} &= 0, & a_{22} &= I_2,\\ c_{11} &= k + k_1, & c_{12} &= -k_1, & c_{22} &= k + k_1. \end{aligned}$$

Substituting these values into the frequency equation (147), we obtain

$$(k + k_1 - I_1p^2)(k + k_1 - I_2p^2) - k_1^2 = 0,$$

from which the two principal frequencies are

$$p_{1,2}^2 = \frac{(I_1 + I_2)(k + k_1)}{2I_1I_2} \mp \sqrt{\frac{(I_1 + I_2)^2(k + k_1)^2}{4I_1^2I_2^2} - \frac{(k^2 + 2kk_1)}{I_1I_2}}. \tag{p}$$

The amplitude ratios, from Eqs. (148), are

$$\left.\begin{aligned} \frac{A_1}{B_1} &= -\frac{k_1}{I_1p_1^2 - (k + k_1)},\\ \frac{A_2}{B_2} &= -\frac{k_1}{I_1p_2^2 - (k + k_1)}. \end{aligned}\right\} \tag{q}$$

The general solution is

$$\phi_1 = A_1\sin(p_1t + \alpha_1) + A_2\sin(p_2t + \alpha_2),$$
$$\phi_2 = -\frac{A_1[I_1p_1^2 - (k + k_1)]}{k_1}\sin(p_1t + \alpha_1) - \frac{A_2[I_1p_2^2 - (k + k_1)]}{k_1}\sin(p_2t + \alpha_2).$$

Since the initial velocities of the bars vanish, we take $\alpha_1 = \alpha_2 = \pi/2$; then

$$\left.\begin{aligned}\phi_1 &= A_1\cos p_1t + A_2\cos p_2t,\\ \phi_2 &= -\frac{A_1[I_1p_1^2 - (k + k_1)]}{k_1}\cos p_1t - \frac{A_2[I_1p_2^2 - (k + k_1)]}{k_1}\cos p_2t.\end{aligned}\right\} \quad (r)$$

Substituting $t = 0$, we obtain A_1 and A_2 from the equations

$$(\phi_1)_0 = A_1 + A_2,$$
$$(\phi_2)_0 = -\frac{A_1}{k_1}[I_1p_1^2 - (k + k_1)] - \frac{A_2}{k_1}[I_1p_2^2 - (k + k_1)].$$

In the particular case where $I_1 = I_2 = I$, Eq. (p) gives the roots

$$p_1^2 = \frac{k}{I}, \qquad p_2^2 = \frac{k + 2k_1}{I}. \quad (p')$$

Then the amplitude ratios (q) become

$$\frac{A_1}{B_1} = 1, \qquad \frac{A_2}{B_2} = -1, \quad (q')$$

and the general solution (r) reduces to

$$\left.\begin{aligned}\phi_1 &= A_1\cos p_1t + A_2\cos p_2t,\\ \phi_2 &= A_1\cos p_1t - A_2\cos p_2t.\end{aligned}\right\} \quad (r')$$

The corresponding two principal modes of vibration are illustrated in Fig. 211*b* and *c*.

By taking k_1 small in comparison with k, the two principal frequencies (p') will be very nearly, but not quite, equal and we obtain the phenomenon of *beating*, as discussed in Art. 6. In such case, as we see from Eqs. (r'), the maximum amplitude of ϕ_1 corresponds to the minimum of ϕ_2 and vice versa, so that we obtain a transfer of energy from one bar to the other. As the oscillations of the bar 1 build up to a maximum, the oscillations of the bar 2 subside and vice versa.

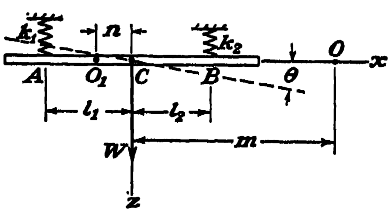

FIG. 212.

Example: Investigate the vibrations in a vertical plane of a bar AB suspended on two springs with constants k_1 and k_2 as shown in Fig. 212. The position of the center of gravity C of the bar is defined by the distances l_1 and l_2, and we assume that horizontal motion of the bar is prevented.

Solution: For generalized coordinates, we take the vertical displacement z of the center of gravity C of the bar from its position of equilibrium and the angle of rotation θ with respect to the same position. Denoting by W the weight of the bar and by i its radius of gyration with respect to the centroidal axis perpendicular to the xz-plane, we obtain

$$\left.\begin{aligned}T &= \frac{1}{2g}(W\dot{z}^2 + Wi^2\dot{\theta}^2),\\ V &= \frac{k_1}{2}(z - l_1\theta)^2 + \frac{k_2}{2}(z + l_2\theta)^2.\end{aligned}\right\} \quad (s)$$

Introducing the notations

$$\frac{(k_1 + k_2)g}{W} = a, \qquad \frac{(-k_1 l_1 + k_2 l_2)g}{W} = b, \qquad \frac{(l_1^2 k_1 + l_2^2 k_2)g}{W} = c,$$

we obtain the following equations of motion:

$$\left.\begin{aligned} \ddot{z} + az + b\theta &= 0, \\ \ddot{\theta} + \frac{b}{i^2} z + \frac{c}{i^2}\theta &= 0. \end{aligned}\right\} \tag{t}$$

The frequency equation is

$$(a - p^2)\left(\frac{c}{i^2} - p^2\right) - \frac{b^2}{i^2} = 0,$$

and the two principal frequencies are

$$p_{1,2}^2 = \frac{1}{2}\left(\frac{c}{i^2} + a\right) \mp \sqrt{\frac{1}{4}\left(\frac{c}{i^2} - a\right)^2 + \frac{b^2}{i^2}}. \tag{u}$$

Assuming that $b > 0$, we find that the lower mode of vibration consists of rotation of the bar with respect to a point O at the distance m from the center of gravity. This distance is obtained from the formula

$$m = \frac{b}{\frac{1}{2}\left(\frac{c}{i^2} - a\right) - \sqrt{\frac{1}{4}\left(\frac{c}{i^2} - a\right)^2 + \frac{b^2}{i^2}}}.$$

The higher mode of vibration consists of rotation with respect to a point O_1 between A and B. The position of this point is defined by the distance $n = i^2/m$.

If $b = 0$, we obtain $m = \infty$ and $n = 1$. This indicates that the lower mode of vibration consists of pure translatory motion of the bar while the higher mode consists of rotation about the center of gravity.

If not only b but also $(c/i^2) - a$ becomes equal to zero, the two principal frequencies become equal.

Example: A horizontal prismatic beam AB carries two masses m_1 and m_2 as shown in Fig. 213. Investigate lateral oscillations of these masses in the plane of the figure, neglecting the mass of the beam.

Solution: We take as coordinates in this case the vertical deflections y_1 and y_2 of the masses m_1 and m_2 from their positions of equilibrium. The kinetic energy of the system then is

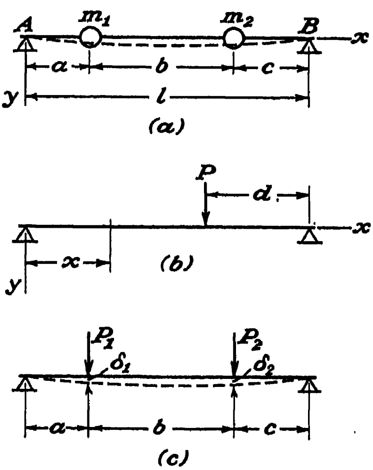

FIG. 213.

$$T = \tfrac{1}{2}(m_1\dot{y}_1^2 + m_2\dot{y}_2^2). \tag{v}$$

In calculating the potential energy of the system, we refer to Fig. 213*b* and use the known equation of the deflection curve

$$y = \frac{P\,dx}{6lEI}(l^2 - d^2 - x^2),$$

for $x < l - d$. From this expression, we find for the statical deflections δ_1 and δ_2 under the loads P_1 and P_2 in Fig. 213c

$$\delta_1 = \frac{P_1a^2(b+c)^2}{3lEI} + \frac{P_2ac}{6lEI}(l^2 - c^2 - a^2),$$
$$\delta_2 = \frac{P_2c^2(a+b)^2}{3lEI} + \frac{P_1ac}{6lEI}(l^2 - c^2 - a^2).$$

Solving these equations for P_1 and P_2, we obtain

$$\left.\begin{aligned} P_1 &= \alpha\delta_1 + \beta\delta_2, \\ P_2 &= \beta\delta_1 + \gamma\delta_2, \end{aligned}\right\} \tag{w}$$

where α, β, γ are constants that can be calculated in any particular case. Using expressions (w), we conclude that the strain energy of bending, equal to the work of the applied loads, is

$$V_1 = \tfrac{1}{2}(P_1\delta_1 + P_2\delta_2) = \tfrac{1}{2}(\alpha\delta_1^2 + 2\beta\delta_1\delta_2 + \gamma\delta_2^2).$$

Returning now to the problem of vibration (Fig. 213a), we conclude from the above discussion that the total potential energy of the system with respect to its equilibrium configuration will be

$$V = \tfrac{1}{2}(\alpha y_1^2 + 2\beta y_1y_2 + \gamma y_2^2). \tag{x}$$

Using expressions (v) and (x) for kinetic and potential energy, we obtain the equations of motion

$$\left.\begin{aligned} m_1\ddot{y}_1 + \alpha y_1 + \beta y_2 &= 0, \\ m_2\ddot{y}_2 + \beta y_1 + \gamma y_2 &= 0. \end{aligned}\right\} \tag{y}$$

and the frequency equation (147) becomes

$$(\alpha - m_1p^2)(\gamma - m_2p^2) - \beta^2 = 0,$$

from which

$$p_{1,2}^2 = \frac{\gamma m_1 + \alpha m_2}{2m_1m_2} \mp \sqrt{\left(\frac{\gamma m_1 + \alpha m_2}{2m_1m_2}\right)^2 - \frac{(\alpha\gamma - \beta^2)}{m_1m_2}}. \tag{z}$$

In the lower mode of vibration, both masses move up and down simultaneously. In the higher mode, one of the masses moves up while the other moves down.

PROBLEMS

136. Two identical simple pendulums of length l are connected by a spring of constant k as shown in Fig. 214. Neglecting the masses of the bars and of the spring, find the principal frequencies p_1 and p_2 and the corresponding amplitude ratios for small oscillations. *Ans.* $p_1 = \sqrt{\frac{g}{l}}$; $p_2 = \sqrt{\frac{g}{l} + \frac{2kh^2}{ml^2}}$; $\left(\frac{\phi_1}{\phi_2}\right)_1 = 1$; $\left(\frac{\phi_1}{\phi_2}\right)_2 = -1$.

137. Two identical bars of weights W and lengths $2a$ are supported at their midpoints by fulcrums A and B and at their ends C and D by springs of constants k and k_1 as shown in Fig. 215. Determine the roots p_1^2 and p_2^2 of the frequency equation (147) and the corresponding amplitude ratios μ_1 and μ_2 as defined by Eqs. (148) for small oscillations in the plane of the figure. Take, as generalized coordinates, the angles of rotation ϕ_1 and ϕ_2 of the bars about the fulcrums.

Ans. $p_{1,2}^2 = (3g/2W)\,[(k + 2k_1) \mp \sqrt{k^2 + 4k_1^2}]$.

138. Referring to Fig. 212 and assuming $W = 966$ lb., $i^2 = 13$ ft.,^{2} $l_1 = 4$ ft.,

$l_2 = 5$ ft., $k_1 = 1{,}600$ lb. per ft., and $k_2 = 2{,}400$ lb. per ft., find the frequencies f_1 and f_2 of the two principal modes of vibration. *Ans.* $f_1 \approx 1.67$; $f_2 = 2.49$.

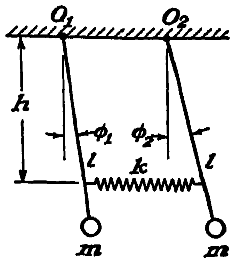

FIG. 214.

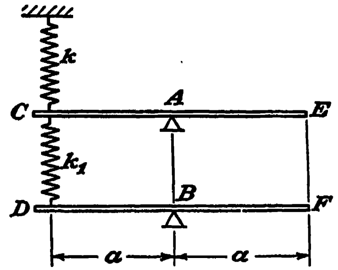

FIG. 215.

139. A square plate of mass m and dimensions $2e$ by $2e$ is attached to the end of a thin cantilever strip of length l and flexural rigidity EI and vibrates in a horizontal plane as shown in Fig. 216. Taking, as coordinates, the displacement y_b of point B and the angle of rotation ϕ of the plate, as shown, write the equations of motion for small vibrations and determine the roots p_1^2 and p_2^2 of the frequency equation.

$$Ans.\quad p_{1,2}^2 = \frac{6EI}{ml^3}\,\frac{1}{1 + 3\frac{e}{l} + 5\frac{e^2}{l^2} \pm \sqrt{\left(1 + 3\frac{e}{l} + 5\frac{e^2}{l^2}\right)^2 - 2\frac{e^2}{l^2}}}.$$

140. Referring to Fig. 213a and assuming that $a = b = c = l/3$ while $m_1 = m_2 = m$, evaluate the constants α, β, γ in Eqs. (w) and the roots p_1^2 and p_2^2 of the frequency equation as given by expressions (z). The flexural rigidity of the beam is EI.

Ans. $p_1^2 = (\alpha + \beta)/m = 486EI/15l^3m$, $p_2^2 = (\alpha - \beta)/m = 486EI/l^3m$.

36. Forced Vibrations of Systems with Two Degrees of Freedom.—In discussing forced vibrations of systems with two degrees of freedom, we begin with the example of two coupled masses (Fig. 207), the free vibrations of which were considered in Art. 34. Assume, now, that in addition to the elastic forces of the springs, there acts on the mass m_1 a horizontal disturbing force $F \sin \omega t$ having angular frequency ω. Using in this case the Lagrangian equation (137) and introducing the notation $F/m_1 = f$ in addition to notations (c) on page 255, we obtain the following equations of motion:

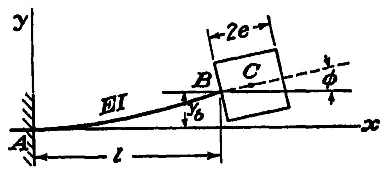

FIG. 216.

$$\left.\begin{aligned}\ddot{x}_1 &= -ax_1 + bx_2 + f\sin\omega t,\\ \ddot{x}_2 &= cx_1 - cx_2.\end{aligned}\right\}\qquad (a)$$

A particular solution of these equations can be taken in the form

$$x_1 = C\sin\omega t, \qquad x_2 = D\sin\omega t. \qquad (b)$$

To obtain the amplitudes C and D, we substitute expressions (b) into Eqs.

(a), which give

$$-C\omega^2 = -aC + bD + f, \qquad -D\omega^2 = cC - cD.$$

Solving these equations for C and D, we obtain

$$C = \frac{f(c - \omega^2)}{(a - \omega^2)(c - \omega^2) - bc}, \qquad D = \frac{fc}{(a - \omega^2)(c - \omega^2) - bc}. \tag{c}$$

With these values of the constants, expressions (b) represent the particular solution of Eqs. (a). They represent, as we see, simple harmonic motions of the masses m_1 and m_2. Such motion of the system is called *forced vibration*. Its frequency is the same as the frequency of the disturbing force, and its amplitudes are determined by Eqs. (c). For very small values of ω, *i.e.*, for a disturbing force of very large period, we can neglect ω^2 in comparison with a and c in expressions (c) and take

$$C = D = \frac{f}{a - b} = \frac{F}{k_1} = \lambda_{st}. \tag{d}$$

The amplitudes of forced vibrations are equal to the deflection of the mass m_1 under the statical action of the force F. This shows that displacements produced by a slowly varying disturbing force can be found with good accuracy from purely statical considerations of the system.

Comparing the denominator in expressions (c) with the left-hand side of the frequency equation (g) on page 255, we conclude that this denominator becomes equal to zero when $\omega^2 = p_1^2$ or when $\omega^2 = p_2^2$. This indicates that the amplitudes of forced vibration grow indefinitely and become very large when the frequency of the disturbing force approaches one of the principal frequencies of the system. These are known as conditions of resonance.

The ratio of the amplitudes C and D of forced vibration from expressions (c) is

$$\frac{C}{D} = \frac{c - \omega^2}{c}. \tag{e}$$

When ω^2 approaches either the value p_1^2 or p_2^2, this ratio approaches the values A_1/B_1 and A_2/B_2, given by expressions (k) and (l) on page 256. This indicates that for a condition of resonance, the forced vibration is identical with the corresponding mode of free vibration as illustrated in Fig. 208.

To see more clearly how the kind of forced vibration depends on the frequency of the disturbing force, let us consider a particular case in which $k_1 = k_2$ and $m_1 = 2m_2$. In such case, $a = c = 2p_0^2$ and $b = p_0^2$, where $p_0 = \sqrt{k_1/m_1}$ is the frequency of the mass m_1 if the mass m_2 in

Fig. 207 is removed. Equations (i), page 255, then give

$$p_1^2 = 0.586p_0^2, \qquad p_2^2 = 3.414p_0^2;$$

and using notation (d), the amplitudes (c) become

$$C = \frac{\lambda_{st}(a-b)\left(1 - \frac{\omega^2}{c}\right)}{a\left(1 - \frac{\omega^2}{a}\right)\left(1 - \frac{\omega^2}{c}\right) - b} = \frac{\lambda_{st}\left(1 - \frac{\omega^2}{2p_0^2}\right)}{2\left(1 - \frac{\omega^2}{2p_0^2}\right)^2 - 1} = \alpha\lambda_{st}, \qquad (f)$$

$$D = \frac{\lambda_{st}}{2\left(1 - \frac{\omega^2}{2p_0^2}\right)^2 - 1} = \beta\lambda_{st}, \qquad (g)$$

where α and β are factors depending only on the ratio ω/p_0. In Fig. 217, these factors are represented by curves. It is seen that when ω

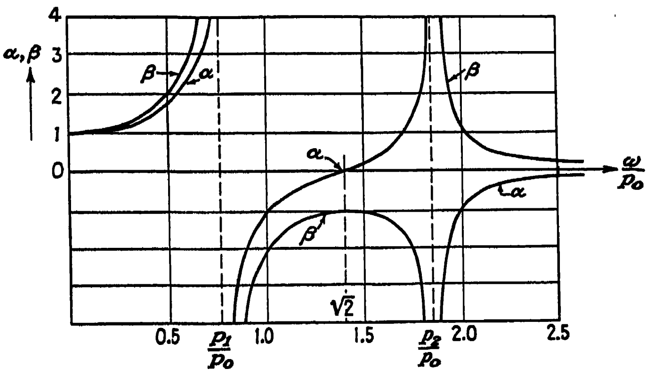

FIG. 217.

approaches zero, α and β approach unity and the amplitudes of the masses m_1 and m_2 approach the value λ_{st}. As ω increases, both amplitudes increase also and grow to infinity as ω approaches the lower principal frequency p_1. The signs of both amplitudes are positive, which indicates that the motions of both masses are *in phase* with the disturbing force. When the frequency of the disturbing force becomes slightly larger than p_1, both amplitudes are large and have negative signs, which indicates that between the disturbing force and the motion of the masses there is a phase difference equal to π. The masses have their maximum displacements to the left when the disturbing force reaches its maximum value in the direction to the right and vice versa. With further increase of ω, both amplitudes are diminishing; and when $\omega = \sqrt{c} = \sqrt{2}\,p_0$, the amplitude C vanishes. When ω becomes larger than $\sqrt{2}\,p_0$, the amplitude C becomes positive while D continues to be negative, which indi-

cates that motion of the mass m_1 again comes into phase with the disturbing force while m_2 is moving with a phase difference π. Finally, as ω approaches the value of the higher principal frequency p_2, both amplitudes grow indefinitely. Beyond that frequency, the mass m_2 is moving in phase with the disturbing force, while m_1 is moving with a phase difference π, and the amplitudes diminish and approach zero as ω continues to increase.

The fact that under certain conditions, the amplitude of the mass m_1 vanishes is of great practical importance. Often in practical applications, we encounter the case of a mass attached to a solid foundation by flexible members and acted upon by a periodic disturbing force, which may produce undesirable forced vibrations. The preceding discussion indicates that these undesirable vibrations can be eliminated by incorporating in the system an additional mass suspended on a properly selected spring. Take, for example the case of a rotor supported by a beam as shown in Fig. 218. Due to unavoidable lack of balance, a centrifugal force F will be set up as shown. Considering only the vertical component of this force[1] and measuring the angle of rotation as shown, we obtain a system with one degree of freedom (a mass m_1 supported by an elastic beam which can be considered as a spring with constant k_1) and acted upon by a vertical pulsating force $F \sin \omega t$. If the rotor always has the same angular velocity ω, we can eliminate forced vibrations of the principal mass m_1 by attaching to the beam an auxiliary mass m_2 and using a spring with constant k_2 of such magnitude that

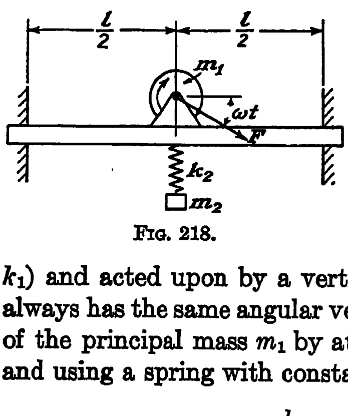

FIG. 218.

$$\frac{k_2}{m_2} = 2p_0^2 = 2\frac{k_1}{m_1} = \omega^2. \tag{h}$$

This follows from the fact that by attaching the mass m_2 we obtain a system with two degrees of freedom to which our above discussion is applicable, and we conclude that the amplitude C of vertical vibrations of the rotor vanishes in virtue of Eqs. (f) and (h). At the same time, as we see from Fig. 217, β and D become negative. The mass m_2 moves with the phase difference π with respect to the disturbing force $F \sin \omega t$ and produces its maximum downward force on the beam when the centrifugal force F is directed upward. In this way, it counteracts the centrifugal force and eliminates vertical vibrations of the rotor. This device is

[1] The horizontal component $F \cos \omega t$ will produce vibration of the beam with an inflection point at the middle, which will not affect vertical vibrations of the rotor.

called a *dynamical damper*. For simplification, friction was omitted in our discussion. In actual cases, it must be taken into consideration, which complicates the problem. The effect of friction will be considered later (see Art. 38).

In the previous discussion, we assumed the disturbing force proportional to $\sin \omega t$. The same conclusions will be reached in the more general case where the force is $F \sin(\omega t + \epsilon)$. We again find that there are two specific values $\omega = p_1$ and $\omega = p_2$ of the impressed frequency at which the amplitudes of forced vibration become very large. For the system in Fig. 218, the angular velocities $\omega = p_1$ and $\omega = p_2$ of the rotor are called *critical speeds*. At these speeds, we obtain a resonance condition and the amplitudes of forced vibration become large.

In the case of a reciprocating engine, the disturbing force, due to unbalance, is of a more complicated nature and can be represented by a series

$$F_1 \sin(\omega t + \epsilon_1) + F_2 \sin(2\omega t + \epsilon_2) + \cdots . \qquad (i)$$

We obtain then a resonance condition each time the angular frequency $i\omega$ of any term of the series (i) coincides with one of the principal frequencies p_1 or p_2 of the system.

In the preceding discussion, we considered only a particular solution of Eqs. (a). To get the complete solution, the free vibrations of the system, represented by Eqs. (m) and (n) on page 256, must be superimposed on the forced vibrations. The constants A_1, A_2, α_1, and α_2, appearing in the expressions for free vibrations, must then be selected so as to satisfy the initial conditions of the system.

As another example, let us consider the system shown in Fig. 219, consisting of a mass m_1 on a spring with constant k_1 and sliding without friction along a smooth horizontal plane. Attached to the mass m_1 is a simple pendulum of mass m_2 and length l as shown. If a horizontal disturbing force $F \sin \omega t$ acts on the mass m_1, forced vibrations of the system will be produced. In investigating the nature of these vibrations, we take, as coordinates, the displacement x of the mass m_1 from its position of equilibrium and the angle θ that the pendulum makes with the vertical. Then, for small vibrations, the kinetic and potential energies of the system are

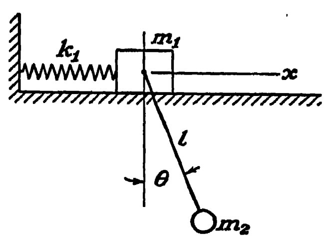

FIG. 219.

$$T = \tfrac{1}{2}[m_1\dot{x}^2 + m_2(\dot{x} + l\dot{\theta})^2],$$
$$V = \tfrac{1}{2}k_1x^2 + \tfrac{1}{2}m_2gl\theta^2,$$

and we obtain the following equations of motion:

$$(m_1 + m_2)\ddot{x} + lm_2\ddot{\theta} + k_1x = F \sin \omega t,$$
$$m_2(\ddot{x} + l\ddot{\theta}) + m_2g\theta = 0,$$

or

$$m_1\ddot{x} + k_1x - m_2g\theta = F \sin \omega t,$$
$$\ddot{x} + l\ddot{\theta} + g\theta = 0.$$

Using notations

$$\frac{k_1}{m_1} = a, \qquad \frac{m_2}{m_1} = b, \qquad \frac{F}{m_1} = f,$$

we obtain

$$\ddot{x} + ax - bg\theta = f \sin \omega t, \qquad \ddot{x} + l\ddot{\theta} + g\theta = 0.$$

Taking a particular solution of these equations in the form

$$x = C \sin \omega t, \qquad \theta = D \sin \omega t$$

and substituting back, we find

$$C = \frac{f(g - l\omega^2)}{(a - \omega^2)(g - l\omega^2) - bg\omega^2}, \qquad D = \frac{f\omega^2}{(a - \omega^2)(g - l\omega^2) - bg\omega^2}. \qquad (j)$$

Critical values of the angular frequency ω are given by the equation

$$(a - \omega^2)(g - l\omega^2) - bg\omega^2 = 0.$$

From the first of expressions (j), we see that the pendulum works as a damper and vibrations of the mass m_1 vanish when

$$g - l\omega^2 = 0,$$

which gives

$$\omega = \sqrt{\frac{g}{l}};$$

i.e., the pendulum must have, for its natural frequency of vibration, the frequency of the pulsating force.

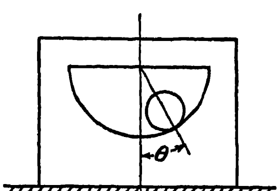

FIG. 220.

Naturally, the use of a simple pendulum as a damper is impractical, but we can take, instead of a pendulum, a roller moving on a cylindrical concave surface inside the mass m_1 as shown in Fig. 220.

As a next example, consider the system shown in Fig. 221*a*. This consists of a rigid bedplate supported by flexible columns and carrying, in bearings A and B, a shaft on which a disk is mounted with small eccentricity e. Let us investigate now the forced vibrations of this system due to rotation of the unbalanced disk with constant angular velocity ω. We assume that the middle plane xy of the disk is a plane of symmetry of the structure and consider only motion of the system in this plane. Referring to Fig. 221*c*, where O represents the equilibrium position of the axis AB, we denote by ξ the horizontal displacement of the bedplate due

to bending of the columns and by ρ the deflection of the shaft during vibration. Thus point E represents the point of intersection of the deflected axis of the shaft with the xy-plane, and C represents an instantaneous position of the center of gravity of the disk. In this way, the configuration of the system is defined by the deflection ξ of the bedplate,

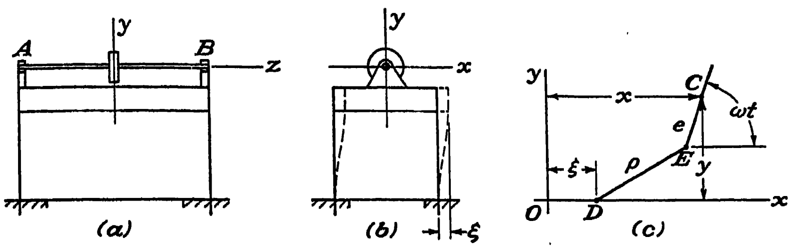

FIG. 221.

the coordinates x and y of the center of gravity C of the disk, and the angle of rotation $\phi = \omega t$ of the disk. Denoting by m and by m_1 the masses of the bedplate and of the disk, respectively, and by I the moment of inertia of the disk about the axis of the shaft, we obtain

$$T = \tfrac{1}{2}[m\dot{\xi}^2 + m_1(\dot{x}^2 + \dot{y}^2) + I\omega^2].$$

In calculating the potential energy of the system, we denote by k and k_1 the spring constants corresponding to deflections ξ of the columns and deflection ρ of the shaft, respectively. Then[1]

$$V = \tfrac{1}{2}(k\xi^2 + k_1\rho^2),$$

where, from Fig. 221c,

$$\rho^2 = (x - \xi - e\cos\omega t)^2 + (y - e\sin\omega t)^2.$$

The Lagrangian equations give now[2]

$$\left.\begin{aligned} m\ddot{\xi} + (k + k_1)\xi - k_1x &= -k_1e\cos\omega t,\\ m_1\ddot{x} + k_1(x - \xi) &= k_1e\cos\omega t,\\ m_1\ddot{y} + k_1y &= k_1e\sin\omega t. \end{aligned}\right\} \qquad (k)$$

The third of these equations contains only the coordinate y, which indicates that vertical vibrations of the disk are of the same kind as for a system with one degree of freedom; the angular frequency of these vibrations is $p_1 = \sqrt{k_1/m_1}$, and the resonance condition occurs when $\omega = p_1$.

The first two of Eqs. (k) define vibrations in the horizontal direction

[1] The potential energy of the gravity force is neglected in this discussion.

[2] The fourth equation, corresponding to the coordinate $\phi = \omega t$ is omitted here. It gives only the magnitude of the torque that must be applied to the shaft to produce the assumed uniform angular velocity ω.

and can be treated as those of a system with two degrees of freedom. Taking a particular solution of these equations in the form

$$x = A \cos \omega t, \qquad \xi = B \cos \omega t$$

and substituting into the equations of motion (k), we obtain

$$\begin{aligned}(-m_1\omega^2 + k_1)A - k_1 B &= ek_1,\\ -k_1 A + (-m\omega^2 + k + k_1)B &= -ek_1.\end{aligned}$$

The frequency equation is

$$(-m_1\omega^2 + k_1)(-m\omega^2 + k + k_1) - k_1^2 = 0. \tag{l}$$

Denoting the left-hand side of this equation by Δ, the amplitudes of forced vibrations will be

$$\begin{aligned}A &= [ek_1(-m\omega^2 + k + k_1) - ek_1^2]:\Delta,\\ B &= [-ek_1(-m_1\omega^2 + k_1) + ek_1^2]:\Delta.\end{aligned}$$

The resonance conditions are obtained when ω coincides with one of the two roots p_2 and p_3 of Eq. (l). When the rigidity of the columns is very large, we put $k = \infty$ and obtain from the first of Eqs. (k) $\xi = 0$. Then the second equation gives

$$m_1\ddot{x} + k_1 x = k_1 e \cos \omega t.$$

This equation then gives, for the critical speed, the value $\omega = p_1$ that we already obtained from the third of Eqs. (k). This is the so-called *whirling speed* of the shaft.

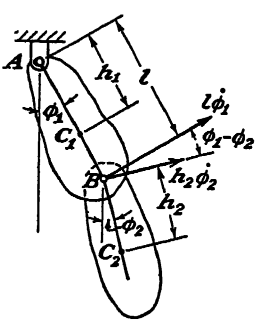

FIG. 222.

As a last example, we shall discuss briefly the forced vibrations of the double pendulum in Fig. 222 if a pulsating couple acts on the body AB. We take as generalized coordinates, in this case, the angles ϕ_1 and ϕ_2 and denote by W_1 and W_2 the weights of the upper and the lower bodies, respectively. Points C_1 and C_2 are the centers of gravity of the two bodies, and we assume that C_1 lies on the line AB. The distances h_1, h_2, and l are as shown in the figure. Let I_1 denote the moment of inertia of the upper body with respect to its axis of rotation through A and I_2, the moment of inertia of the lower body about its gravity axis normal to the plane of the figure. With these notations, the kinetic energy of the upper body is $I_1\dot{\phi}_1^2/2$. The kinetic energy of the lower body consists of the two parts: (1) the kinetic energy due to the velocity v of the center of gravity C_2 and (2) the kinetic energy of rotation with respect to the center of gravity. In this way we obtain $W_2v^2/2g + I_2\dot{\phi}_2^2/2$.

Observing that

$$v^2 = l^2\dot{\phi}_1^2 + h_2^2\dot{\phi}_2^2 + 2h_2l\dot{\phi}_1\dot{\phi}_2\cos(\phi_1 - \phi_2)$$

and taking for small oscillations $\cos(\phi_1 - \phi_2) \approx 1$, we obtain

$$T = \frac{1}{2}\left[\left(I_1 + \frac{W_2l^2}{g}\right)\dot{\phi}_1^2 + \left(I_2 + \frac{h_2^2W_2}{g}\right)\dot{\phi}_2^2 + 2h_2l\frac{W_2}{g}\dot{\phi}_1\dot{\phi}_2\right].$$

The potential energy is due to gravity forces and for small oscillation is

$$V = \tfrac{1}{2}[(W_1h_1 + W_2l)\phi_1^2 + W_2h_2\phi_2^2].$$

Comparing these values of V and T with expressions (145) (see page 259), we conclude that

$$c_{11} = W_1h_1 + W_2l, \qquad c_{22} = W_2h_2, \qquad c_{12} = 0,$$
$$a_{11} = I_1 + \frac{W_2l^2}{g}, \qquad a_{22} = I_2 + \frac{h_2^2W_2}{g}, \qquad a_{12} = h_2l\frac{W_2}{g}.$$

Then the frequency equation (147) gives

$$p_{1,2}^2 = \frac{a_{11}c_{22} + c_{11}a_{22} \mp \sqrt{(a_{11}c_{22} - c_{11}a_{22})^2 + 4a_{12}^2c_{11}c_{22}}}{2(a_{11}a_{22} - a_{12}^2)}. \tag{m}$$

Considering the fundamental mode of vibration with frequency p_1, we find from Eqs. (148) that the ratio of amplitudes of the angles of oscillation ϕ_1 and ϕ_2 is

$$\frac{\phi_1}{\phi_2} = \frac{a_{12}p_1^2}{c_{11} - a_{11}p_1^2} = \frac{c_{22} - a_{22}p_1^2}{a_{12}p_1^2}. \tag{n}$$

The same ratio can also be put in the following form

$$\frac{\phi_1}{\phi_2} = \frac{c_{22}a_{12}}{c_{11}a_{22} - (a_{11}a_{22} - a_{12}^2)p_1^2}.$$

Substituting for p_1^2 its value from expression (m), we obtain

$$\frac{\phi_1}{\phi_2} = \frac{2a_{12}c_{22}}{c_{11}a_{22} - a_{11}c_{22} + \sqrt{(a_{11}c_{22} - c_{11}a_{22})^2 + 4a_{12}^2c_{11}c_{22}}}. \tag{o}$$

Of some practical interest is the case in which

$$c_{11}a_{22} - a_{11}c_{22} = 0. \tag{p}$$

Then, for the fundamental mode of vibration, we obtain

$$\frac{\phi_1}{\phi_2} = \sqrt{\frac{c_{22}}{c_{11}}}.$$

If, in addition, we have

$$c_{11} = c_{22}, \tag{q}$$

the angles ϕ_1 and ϕ_2 remain equal during the lower mode of vibration. A bell and its clapper can be considered as a double pendulum; and if conditions (p) and (q) are satisfied, the clapper cannot strike the bell if the system performs the fundamental mode of vibration. If a pulsating force in resonance with the fundamental mode of vibration is applied to the bell, the amplitude ratio of forced vibration is the same as that of the fundamental mode and the bell will not ring.

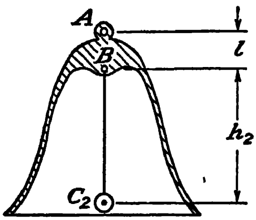

FIG. 223.

To see the physical significance of this phenomenon of a "silent bell," let us consider the case shown in Fig. 223 where, for simplicity, we treat the clapper as a simple pendulum of weight W_2 and length h_2. Then following the notations for Fig. 222, we have $I_2 = 0$ and conditions (p) and (q) become

$$\left.\begin{aligned}(W_1h_1 + W_2l)\frac{W_2}{g}h_2^2 - \left(I_1 + \frac{W_2}{g}l^2\right)W_2h_2 &= 0,\\ W_1h_1 + W_2l &= W_2h_2.\end{aligned}\right\} \qquad (q')$$

By virtue of the second of these conditions, the first reduces to

$$\frac{W_2}{g}(h_2^2 - l^2) = I_1 = \frac{W_1}{g}(h_1^2 + i_1^2),$$

where i_1 is the radius of gyration of the bell about its gravity axis normal to the plane of the figure. Using this equation together with the second of Eqs. (q'), we obtain

$$l + h_2 = h_1 + \frac{i_1^2}{h_1}. \qquad (r)$$

We see that the left-hand side of this equation represents the position of the center of gravity C_2 of the clapper below point A while the right-hand side represents the position of the center of oscillation[1] of the bell below A. Thus the bell is silent if the center of gravity of the clapper and the center of oscillation of the bell coincide.

PROBLEMS

141. Referring to the system in Fig. 207, page 254, and using the numerical data given in the example on page 258, calculate the amplitudes of forced vibration produced by a horizontal disturbing force $10 \sin(10\pi t)$ lb. applied to the mass m_1.

Ans. $C = 0.087$ in.; $D = 0.117$ in.

142. Solve the preceding problem assuming that the disturbing force acts on the mass m_2.

Ans. $C = 0.015$ in.; $D = 0.037$ in.

[1] See the authors' "Engineering Mechanics," 2d ed., p. 401.

37. Vibrations with Viscous Damping.—In discussing vibrations up to now, friction forces were neglected. As a result of this, we found that amplitudes of free vibration remain constant while amplitudes of forced vibration grow indefinitely in conditions of resonance. In actual cases, we always have some friction forces; and owing to their action, the energy of a vibrating system will be dissipated; the amplitudes of free vibration will be gradually damped out, and the amplitudes of forced vibration will not grow indefinitely but will approach certain limiting values depending on the amount of friction. In discussing systems with one degree of freedom, we already have seen that the problem can be greatly simplified if the friction forces are proportional to the velocities of the moving parts of the system. The same conclusion holds also for systems with several degrees of freedom.

The Lagrangian equations can be generalized so as to take care of viscous friction. For this purpose, it is advantageous to introduce the notion of the rate at which energy is dissipated. Considering first a single particle moving along the x-axis, we may take the resisting force due to viscous damping equal to $-c\dot{x}$, where the minus sign indicates that the force always opposes motion and the constant coefficient c represents the magnitude of the friction force when the velocity is unity. The work done by the friction force during a small displacement δx is $-c\dot{x}\,\delta x$, and the amount of energy dissipated is

$$c\dot{x}\,\delta x = c\dot{x}^2\delta t,$$

so that the time rate at which energy is dissipated in this case is $c\dot{x}^2$. Now we introduce the dissipation function F, defined by the equation

$$F = \tfrac{1}{2}c\dot{x}^2 \tag{a}$$

and representing half the rate at which energy is dissipated. The friction force is then obtained by differentiation, namely:

$$-c\dot{x} = -\frac{dF}{d\dot{x}}.$$

In the general case of motion of a particle, the velocity can be resolved into three orthogonal components and the dissipation function is

$$F = \tfrac{1}{2}(c_1\dot{x}^2 + c_2\dot{y}^2 + c_3\dot{z}^2), \tag{b}$$

where the factors c_1, c_2, c_3 define the viscous friction in the x-, y-, and z-directions, respectively.

In the case of a system of particles, the dissipation function is obtained by a summation of expressions (b) for all particles of the system and we have

$$F = \tfrac{1}{2}\Sigma(c_{1i}\dot{x}_i^2 + c_{2i}\dot{y}_i^2 + c_{3i}\dot{z}_i^2). \tag{c}$$

If we are using generalized coordinates $q_1, q_2, \ldots$, the friction force R_i corresponding to any coordinate q_i is obtained by giving to the system a virtual displacement δq_i and writing the expression for virtual work. Thus

$$-\sum\left(c_{1i}\dot{x}_i\frac{\partial x_i}{\partial q_i} + c_{2i}\dot{y}_i\frac{\partial y_i}{\partial q_i} + c_{3i}\dot{z}_i\frac{\partial z_i}{\partial q_i}\right)\delta q_i = R_i\delta q_i.$$

Observing from page 213 that

$$\frac{\partial x_i}{\partial q_i} = \frac{\partial \dot{x}_i}{\partial \dot{q}_i},$$

we obtain

$$R_i = -\sum\left(c_{1i}\dot{x}_i\frac{\partial \dot{x}_i}{\partial \dot{q}_i} + c_{2i}\dot{y}_i\frac{\partial \dot{y}_i}{\partial \dot{q}_i} + c_{3i}\dot{z}_i\frac{\partial \dot{z}_i}{\partial \dot{q}_i}\right) = -\frac{\partial F}{\partial \dot{q}_i}. \tag{d}$$

It is seen that to calculate generalized friction forces, we need only to represent the dissipation function (*c*) in generalized coordinates and make the derivatives (*d*). Substituting for $\dot{x}_i, \dot{y}_i, \dot{z}_i$, in expression (*c*), their expressions through the generalized coordinates (see page 212), we shall obtain for the dissipation function a homogeneous function of second degree in $\dot{q}_1, \dot{q}_2, \ldots$; *i.e.*,

$$F = \tfrac{1}{2}(b_{11}\dot{q}_1^2 + 2b_{12}\dot{q}_1\dot{q}_2 + b_{22}\dot{q}_2^2 + \cdots). \tag{e}$$

The coefficients $b_{11}, b_{12}, \ldots$ generally depend on the configuration of the system; but for the case of small vibrations in the neighborhood of a configuration of stable equilibrium, these coefficients can be treated with good accuracy as being constants, as already discussed in Art. 33.

We assumed above that the friction forces were proportional to the absolute velocities of the particles, but similar conclusions are obtained also if the friction forces are proportional to the relative velocities of the particles. If we have, for example, the friction force $-c_1(\dot{x}_1 - \dot{x}_2)$ acting on a particle 1 and proportional to its velocity relative to a particle 2, there will be an equal and opposite force $-c_1(\dot{x}_2 - \dot{x}_1)$ acting on the particle 2. The corresponding generalized friction force R_i is then obtained from the expression for virtual work. Thus

$$\begin{aligned}
-c_1(\dot{x}_1 - \dot{x}_2)\delta x_1 - c_1(\dot{x}_2 - \dot{x}_1)\delta x_2 &= -c_1(\dot{x}_1 - \dot{x}_2)\delta(x_1 - x_2)\\
&= -c_1(\dot{x}_1 - \dot{x}_2)\frac{\partial(x_1 - x_2)}{\partial q_i}\delta q_i\\
&= -c_1(\dot{x}_1 - \dot{x}_2)\frac{\partial(\dot{x}_1 - \dot{x}_2)}{\partial \dot{q}_i}\delta q_i\\
&= -\frac{1}{2}c_1\frac{\partial(\dot{x}_1 - \dot{x}_2)^2}{\partial \dot{q}_i}\delta q_i,
\end{aligned}$$

from which

$$R_i = -\frac{1}{2}c_1\frac{\partial(\dot{x}_1 - \dot{x}_2)^2}{\partial \dot{q}_i}.$$

If we substitute for the velocities $\dot{x}_1$, $\dot{x}_2$ their expressions through generalized velocities $\dot{q}_1, \dot{q}_2, \ldots$, we shall find again that R_i is obtained as the partial derivative with respect to $\dot{q}_i$ of the dissipation function which is a homogeneous second-degree function of the velocities $\dot{q}_1, \dot{q}_2, \ldots$ similar to the function (e).

As soon as we have the expression for the dissipation function, Lagrangian equations can be written in the form

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_i} - \frac{\partial T}{\partial q_i} + \frac{\partial V}{\partial q_i} + \frac{\partial F}{\partial \dot{q}_i} = Q_i. \tag{149}$$

The generalized force corresponding to any coordinate q_i consists, as we see, of three parts: (1) the part $-\partial V/\partial q_i$, due to forces having potential, such as gravity forces and forces in elastic springs, (2) the part $-\partial F/\partial \dot{q}_i$ due to viscous friction, and (3) the part Q_i, in which are included other forces acting on the system, such as a pulsating disturbing force producing forced vibrations.

In studying free vibrations with damping, Q_i vanishes and Eq. (149) reduces to

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_i} - \frac{\partial T}{\partial q_i} + \frac{\partial V}{\partial q_i} + \frac{\partial F}{\partial \dot{q}_i} = 0. \tag{150}$$

Let us now consider the application of Eq. (150) to the case of a system with two degrees of freedom, assuming that the coordinates q_1 and q_2 are *principal coordinates* (see page 263). Then for small vibrations, the quadratic functions for kinetic energy, potential energy, and dissipation energy are

$$\left.\begin{aligned} T &= \tfrac{1}{2}(a_{11}\dot{q}_1^2 + a_{22}\dot{q}_2^2), \\ V &= \tfrac{1}{2}(c_{11}q_1^2 + c_{22}q_2^2), \\ F &= \tfrac{1}{2}(b_{11}\dot{q}_1^2 + 2b_{12}\dot{q}_1\dot{q}_2 + b_{22}\dot{q}_2^2). \end{aligned}\right\} \tag{f}$$

Using these expressions in the Lagrangian equation (150), we obtain

$$\left.\begin{aligned} a_{11}\ddot{q}_1 + c_{11}q_1 + b_{11}\dot{q}_1 + b_{12}\dot{q}_2 &= 0, \\ a_{22}\ddot{q}_2 + c_{22}q_2 + b_{12}\dot{q}_1 + b_{22}\dot{q}_2 &= 0, \end{aligned}\right\} \tag{g}$$

which are linear equations with constant coefficients. A solution of Eqs. (g) can be taken in the form

$$q_1 = Ce^{st}, \qquad q_2 = De^{st}, \tag{h}$$

where C, D, and s are undetermined constants. Substituting back into

Eqs. (g), we obtain

$$\left.\begin{aligned} C(a_{11}s^2 + b_{11}s + c_{11}) + Db_{12}s &= 0, \\ Cb_{12}s + D(a_{22}s^2 + b_{22}s + c_{22}) &= 0. \end{aligned}\right\} \qquad (i)$$

These equations may give, for C and D, solutions different from zero only if their determinant vanishes, *i.e.*, when we have

$$(a_{11}s^2 + b_{11}s + c_{11})(a_{22}s^2 + b_{22}s + c_{22}) - b_{12}{}^2s^2 = 0. \qquad (j)$$

This is an equation of the fourth degree in s and will have four roots, which we denote by s_1, s_2, s_3, s_4. When damping is not excessive, so that we actually get vibrations, all four roots are complex with negative real parts and we have

$$\left.\begin{aligned} s_1 &= -n_1 + ip_1, \\ s_2 &= -n_1 - ip_1, \\ s_3 &= -n_2 + ip_2, \\ s_4 &= -n_2 - ip_2, \end{aligned}\right\} \qquad (k)$$

where n_1 and n_2 are positive numbers[1] and $i = \sqrt{-1}$. Substituting each of these roots into Eqs. (i), the corresponding ratio C/D will be obtained. Denoting these ratios by $\mu_1 \ldots \mu_4$, we can write the general solution of Eqs. (g) in the following form:

$$\left.\begin{aligned} q_1 &= C_1e^{s_1t} + C_2e^{s_2t} + C_3e^{s_3t} + C_4e^{s_4t}, \\ q_2 &= \mu_1C_1e^{s_1t} + \mu_2C_2e^{s_2t} + \mu_3C_3e^{s_3t} + \mu_4C_4e^{s_4t}. \end{aligned}\right\} \qquad (l)$$

This solution contains four constants of integration $C_1 \ldots C_4$, which can be selected in each particular case so as to satisfy known conditions regarding the initial values of the coordinates q_1 and q_2 and velocities $\dot{q}_1$ and $\dot{q}_2$.

It will be advantageous to introduce now trigonometric, instead of exponential, functions. Taking the first two terms in the expression for q_1, we have

$$\begin{aligned} C_1e^{s_1t} + C_2e^{s_2t} &= \tfrac{1}{2}(C_1 + C_2)(e^{s_1t} + e^{s_2t}) + \tfrac{1}{2}(C_1 - C_2)(e^{s_1t} - e^{s_2t}) \\ &= (C_1 + C_2)e^{-n_1t}\cos p_1t - i(C_1 - C_2)e^{-n_1t}\sin p_1t. \end{aligned}$$

Similarly we can transform the last two terms in the expression for q_1 and also the terms in the expression for q_2. Then introducing new notations for the constants, we can write

$$\left.\begin{aligned} q_1 &= e^{-n_1t}(A_1\cos p_1t + A_2\sin p_1t) + e^{-n_2t}(A_3\cos p_2t + A_4\sin p_2t), \\ q_2 &= e^{-n_1t}(B_1\cos p_1t + B_2\sin p_1t) + e^{-n_2t}(B_3\cos p_2t + B_4\sin p_2t). \end{aligned}\right\} \qquad (m)$$

[1] The proof of this can be found in Riemann-Weber's "Differential-gleichungen der Physik," vol. 1, p. 125, 1925.

Due to the fact that $A_1 \ldots B_4$ are all expressed through $C_1 \ldots C_4$, there will be only four independent constants in the solutions (m) which have to be determined from four initial conditions. We see from Eqs. (m) that the coordinates q_1 and q_2 are obtained by superposition of two damped vibrations having the angular frequencies p_1 and p_2. When viscous friction is small, we find that these frequencies differ from those found in the absence of damping only by small quantities of second order and that we can usually neglect the effect of damping on natural frequencies of vibration. This conclusion was already discussed in the case of systems with one degree of freedom (see page 38).

If we have a system with very large damping, it is possible that two roots or even all four of the roots (k) will become real and negative. Assuming, for example, that the last two roots are real, we shall find, as in the case of systems with one degree of freedom (see page 35), that the corresponding motion is aperiodic and that the complete expression for the motion will consist of damped vibrations superimposed on an aperiodic motion.

In discussing forced vibrations with damping, we must use Eq. (149). As an example, let us consider the system shown in Fig. 207 and assume that a device with viscous friction is incorporated between the masses m_1 and m_2 such that the friction forces acting on these masses are $c(\dot{x}_2 - \dot{x}_1)$ and $c(\dot{x}_1 - \dot{x}_2)$, respectively. Then using the notations of Art. 34 and assuming that a pulsating force $P \cos \omega t$ acts on the mass m_1, Eq. (149) gives the following equations of motion:

$$\left.\begin{aligned} m_1\ddot{x}_1 + k_1x_1 - k_2(x_2 - x_1) &= c(\dot{x}_2 - \dot{x}_1) + P \cos \omega t, \\ m_2\ddot{x}_2 + k_2(x_2 - x_1) &= c(\dot{x}_1 - \dot{x}_2). \end{aligned}\right\} \qquad (n)$$

The complete solution of these equations consists of two parts: (1) free vibrations with damping and (2) forced vibrations. Free vibrations, as we have just seen, are gradually damped out, and we have, practically, to deal with the steady forced vibrations sustained by the pulsating force $P \cos \omega t$. These forced vibrations are obtained as a particular solution of Eqs. (n). Observing that owing to damping, there must be a phase difference between the pulsating force and the motion that it induces, we take this solution in the form

$$\left.\begin{aligned} x_1 &= C_1\cos \omega t + C_2\sin \omega t, \\ x_2 &= C_3\cos \omega t + C_4\sin \omega t. \end{aligned}\right\} \qquad (o)$$

Substituting these expressions into Eqs. (n) and equating to zero the coefficients before $\cos \omega t$ and $\sin \omega t$, we obtain four linear equations for calculating the constants $C_1 \ldots C_4$. Solving these equations and substituting the solutions into Eqs. (o), we shall obtain the required forced vibrations of the system.

In practical applications, we need very often to reduce forced vibrations of the mass m_1 on which the pulsating force is acting, and for this purpose, the mass m_2 is attached simply as a *vibration absorber*. To see the effect of such an absorber on the motion of the mass m_1, let us consider the amplitude λ_1 of this vibration, which is obtained from the equation

$$\lambda_1^2 = (x_1)^2_{\max} = C_1^2 + C_2^2.$$

We shall omit the intermediate calculation of the constants C_1 and C_2 mentioned above and give only the final results[1] in which the following notations are used:

$\lambda_{st} = P/k_1$ = static deflection of mass m_1 produced by force P,
$p = \sqrt{k_1/m_1}$ = angular frequency of mass m_1 (main system) in absence of mass m_2,
$p_1 = \sqrt{k_2/m_2}$ = angular frequency of mass m_2 if attached to a fixed point,
$\beta = m_2/m_1$ = ratio of mass of absorber to that of main system,
$\delta = p_1/p$ = ratio of natural frequencies of absorber and of main system,
$\mu = c/2m_2p$ = viscous damping factor,
$\gamma = \omega/p$ = ratio of frequency of disturbing force to that of natural vibration of main system.

The amplitude λ_1 of forced vibration is then given by the equation

$$\frac{\lambda_1^2}{\lambda_{st}^2} = \frac{4\mu^2\gamma^2 + (\gamma^2 - \delta^2)^2}{4\mu^2\gamma^2(\gamma^2 - 1 + \beta\gamma^2)^2 + [\beta\,\delta^2\gamma^2 - (\gamma^2 - 1)(\gamma^2 - \delta^2)]^2}. \qquad (p)$$

From this expression, the amplitude λ_1 of forced vibration can be calculated for any value of $\gamma = \omega/p$, provided the quantities δ and β, defining the frequency and the mass of the absorber, and also the quantity μ, defining viscous friction, are known.

Let us begin with a discussion of some extreme cases in which the problem is simplified. If we put $\mu = 0$, *i.e.*, if friction is neglected, we obtain from expression (p)

$$\frac{\lambda_1}{\lambda_{st}} = \frac{\gamma^2 - \delta^2}{\beta\,\delta^2\gamma^2 - (\gamma^2 - 1)(\gamma^2 - \delta^2)}. \qquad (q)$$

This expression can be brought into coincidence with the expression for C/λ_{st} obtained for forced vibration without damping [see Eq. (f), page 271]. In Fig. 224, the absolute values of the ratio λ_1/λ_{st} are plotted against $\omega/p = \gamma$ and are given by the dotted lines ($\mu = 0$) for a particular case in which $\beta = \frac{1}{20}$ and $\delta = 1$. The ratio (q) changes sign for

[1] For a more detailed discussion of the problem, see paper by J. Ormondroyd and J. P. Den Hartog, *Trans. Am. Soc. Mech. Eng.*, vol. 50, p. 7, 1928.

$\gamma = 0.895$ and for $\gamma = 1.12$ at which values the system is in a condition of resonance.

Another extreme case is obtained by taking $\mu = \infty$. In this case, there is no relative motion between the masses m_1 and m_2 and we obtain

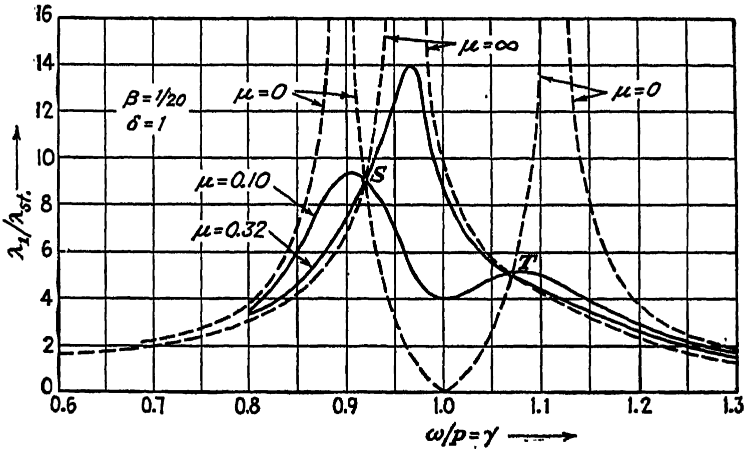

FIG. 224.

a system with one degree of freedom having the mass equal to $m_1 + m_2$ and the spring constant k_1. In this case, expression (c) gives

$$\frac{\lambda_1^2}{\lambda_{st}^2} = \frac{1}{(\gamma^2 - 1 + \beta\gamma^2)^2}. \tag{r}$$

The resonance condition occurs when

$$\gamma^2 - 1 + \beta\gamma^2 = 0,$$

which gives

$$\gamma_{cr} = \sqrt{\frac{1}{1 + \beta}}. \tag{s}$$

The absolute values of λ_1/λ_{st} for our numerical example are given in Fig. 224 by the dotted line marked $\mu = \infty$.

For any intermediate value of μ, the absolute values of λ_1/λ_{st} have to be calculated from expression (p). In Fig. 224, the corresponding curves for $\mu = 0.10$ and $\mu = 0.32$ are shown. The curves for various values of μ all pass through the points S and T where the curves $\mu = 0$ and $\mu = \infty$ intersect. The abscissas of these points are obtained by equating the absolute values of λ_1/λ_{st} obtained from expressions (q) and (r), which give

$$\frac{\gamma^2 - \delta^2}{\beta\,\delta^2\gamma^2 - (\gamma^2 - 1)(\gamma^2 - \delta^2)} = \frac{1}{\gamma^2 - 1 + \beta\gamma^2}. \tag{t}$$

The same equation is also obtained from expression (p), which can be written in the form

$$\frac{M\mu^2 + N}{P\mu^2 + Q},$$

where M, N, P, and Q are expressions that do not contain μ. This expression will be independent of μ^2 if

$$\frac{M}{P} = \frac{N}{Q},$$

and this brings us again to Eq. (t), which gives

$$(\gamma^2 - \delta^2)(\gamma^2 - 1 + \beta\gamma^2) = \beta\delta^2\gamma^2 - (\gamma^2 - 1)(\gamma^2 - \delta^2)$$

or

$$\gamma^4 - 2\gamma^2\frac{1 + \delta^2 + \beta\delta^2}{2 + \beta} + \frac{2\delta^2}{2 + \beta} = 0. \qquad (u)$$

From this quadratic equation in γ^2, two roots γ_1^2 and γ_2^2, which give abscissas of the points S and T, can be readily calculated in each particular case. The ordinates of the same points are now obtained by substituting the roots γ_1^2 and γ_2^2 into Eq. (r). Assuming that γ_1^2 is the smaller root, we obtain the following positive values for the ordinates

$$-\frac{1}{\gamma_1^2 - 1 + \beta\gamma_1^2} \quad \text{and} \quad \frac{1}{\gamma_2^2 - 1 + \beta\gamma_2^2}. \qquad (v)$$

The magnitudes of these ordinates will depend on the quantities β and δ defining the mass m_2 and the spring constant k_2 of the absorber. In practical applications, it is advantageous to select these constants so as to make the ordinates (v) equal.[1] This gives

$$-\frac{1}{\gamma_1^2 - 1 + \beta\gamma_1^2} = \frac{1}{\gamma_2^2 - 1 + \beta\gamma_2^2}, \qquad (w)$$

and we obtain

$$\gamma_1^2 + \gamma_2^2 = \frac{2}{1 + \beta}.$$

At the same time, the sum of the two roots of the quadratic equation (u) must be equal to the coefficient of the middle term with opposite sign. Hence

$$\frac{2}{1 + \beta} = \left(2\,\frac{1 + \delta^2 + \beta\delta^2}{2 + \beta}\right),$$

and we obtain

$$\delta = \frac{1}{1 + \beta}. \qquad (x)$$

[1] See paper by E. Hahnkamm, *Z. angew. Math. Mech.*, vol. 13, p. 183, 1933.

This simple formula gives the proper "tuning" of the absorber. When the mass of the absorber has been selected, β is known and from Eq. (x) we find δ, which defines the spring constant k_2 of the absorber. Substituting expression (x) for δ into Eq. (u), we find

$$\gamma^2 = \frac{1}{1+\beta}\left(1 \pm \sqrt{\frac{\beta}{2+\beta}}\right),$$

and the ordinates (v) become

$$\frac{\lambda_1}{\lambda_{st}} = \sqrt{\frac{2+\beta}{\beta}}. \tag{y}$$

It remains now to select the proper value of μ, which defines the intensity of damping. The damping in the absorber must be such as to make forced vibrations of the main system as small as possible for all possible frequencies of the pulsating force. Calculations show that very satisfactory conditions are realized by taking μ such that the corresponding full line curve in Fig. 224 has a maximum at point S or at point T. In such case, the maximum value of λ_1/λ_{st} differs but little from the ordinate of the points S and T and expression (y) gives that maximum with a fair accuracy.[1]

38. Systems with Several Degrees of Freedom.—The problem of small free vibrations of a system with several degrees of freedom about a configuration of stable equilibrium requires the solution of Eqs. (144), which we repeat here.

$$\left.\begin{array}{l} a_{11}\ddot{q}_1 + a_{12}\ddot{q}_2 + a_{13}\ddot{q}_3 + \cdots + c_{11}q_1 + c_{12}q_2 + c_{13}q_3 + \cdots = 0, \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots, \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots, \\ a_{n1}\ddot{q}_1 + a_{n2}\ddot{q}_2 + a_{n3}\ddot{q}_3 + \cdots + c_{n1}q_1 + c_{n2}q_2 + c_{n3}q_3 + \cdots = 0. \end{array}\right\} \tag{a}$$

In solving these equations, we proceed in the same manner as in the case of systems with two degrees of freedom and take a particular solution in the form

$$q_1 = \lambda_1 \sin(pt+\alpha), \quad q_2 = \lambda_2 \sin(pt+\alpha), \; \cdots, q_n = \lambda_n \sin(pt+\alpha). \tag{b}$$

Substituting in Eqs. (a), we obtain the following system of homogeneous linear equations for calculating the amplitudes $\lambda_1, \lambda_2, \ldots \lambda_n$:

$$\left.\begin{array}{l} \lambda_1(a_{11}p^2 - c_{11}) + \lambda_2(a_{12}p^2 - c_{12}) + \cdots + \lambda_n(a_{1n}p^2 - c_{1n}) = 0, \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots, \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots, \\ \lambda_n(a_{n1}p^2 - c_{n1}) + \lambda_2(a_{n2}p^2 - c_{n2}) + \cdots + \lambda_n(a_{nn}p^2 - c_{nn}) = 0. \end{array}\right\} \tag{c}$$

[1] See paper by Hahnkamm, Schiffbautechnische Gesellschaft, *Versamlung*, Berlin, November, 1935.

These equations can give, for $\lambda_1 \ldots \lambda_n$, solutions different from zero only if their determinant vanishes. In this way we obtain the frequency equation

$$\Delta(p^2) = 0, \tag{d}$$

where the symbol $\Delta(p^2)$ indicates the determinant of Eqs. (c). This equation is of the nth degree in p^2 and has n roots. In the case of vibrations about a position of stable equilibrium, all these roots are real and positive. Let p_s^2 be any one of these roots. Substituting it into Eqs. (c), we can calculate the $(n - 1)$ ratios $\lambda_1/\lambda_2, \lambda_1/\lambda_3, \ldots \lambda_1/\lambda_n$ which define the *mode of vibration* corresponding to the frequency p_s. The corresponding motion will be

$$\left.\begin{aligned} q_1 &= \lambda_1 \sin(p_s t + \alpha_s), \\ q_2 &= \lambda_2 \sin(p_s t + \alpha_s), \\ &\ldots\ldots\ldots\ldots\ldots\ldots, \\ q_n &= \lambda_n \sin(p_s t + \alpha_s). \end{aligned}\right\} \tag{e}$$

All coordinates perform simple harmonic motions of the same frequency p_s and of the same phase α_s. They simultaneously pass through their zero values, corresponding to the equilibrium configuration of the system, and simultaneously reach their extreme values. Since the ratios of the amplitudes $\lambda_1, \lambda_2, \cdots \lambda_n$ are already calculated from Eqs. (c), there remain only two arbitrary constants in the solution (e); namely: α_s and one of the amplitudes, say λ_1. The solution (e) represents one of the *principal* or *natural* vibrations of the system. We shall have as many such vibrations as the number of roots of Eq. (d), *i.e.*, as the number n of the degrees of freedom of the system.

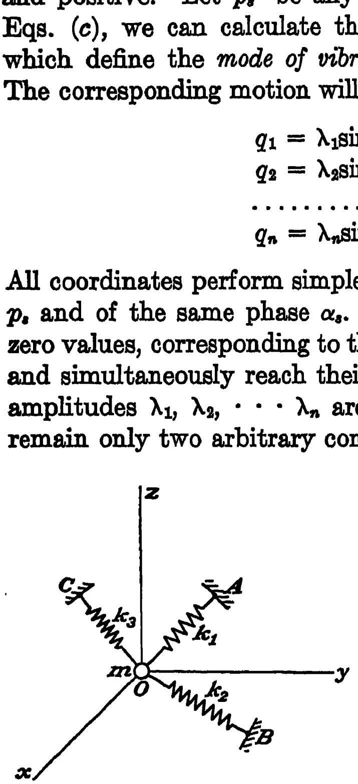

FIG. 225.

The general solution of Eqs. (a) is obtained by superposition of the principal vibrations; and since to each vibration there correspond two arbitrary constants, we shall have altogether $2n$ constants, which always can be selected so as to satisfy $2n$ initial conditions concerning the n initial values of the coordinates and the n corresponding initial velocities.

After this general discussion, let us consider an example. Assume that a particle of mass m is fixed in space by three helical springs the axes of which do not lie in one plane (Fig. 225). It is required to investigate vibrations of the particle if disturbed from its position of equilibrium. In this case, we have a system with three degrees of freedom,

and we take, as coordinates of the system, the rectangular coordinates x,y,z of the particle, measured from the position of equilibrium. Considering them small in comparison with the lengths of the springs, we can readily find the elongations of the springs during vibration. Denoting by α_1, β_1, γ_1 the direction cosines of the axis of the spring OA, the elongation of this spring will be $-(\alpha_1 x + \beta_1 y + \gamma_1 z)$ and the corresponding strain energy[1] is $\frac{1}{2}k_1(\alpha_1 x + \beta_1 y + \gamma_1 z)^2$. Similar expressions, with the necessary change of subscripts, will be obtained for the two other springs, and the total strain energy of the system becomes

$$V = \tfrac{1}{2}k_1(\alpha_1 x + \beta_1 y + \gamma_1 z)^2 + \tfrac{1}{2}k_2(\alpha_2 x + \beta_2 y + \gamma_2 z)^2 + \tfrac{1}{2}k_3(\alpha_3 x + \beta_3 y + \gamma_3 z)^2,$$

which can be rewritten in the following form:

$$V = \tfrac{1}{2}(c_{11}x^2 + c_{22}y^2 + c_{33}z^2 + 2c_{12}xy + 2c_{13}xz + 2c_{23}yz), \tag{f}$$

where

$$\begin{aligned} c_{11} &= k_1\alpha_1^2 + k_2\alpha_2^2 + k_3\alpha_3^2, \\ c_{12} &= k_1\alpha_1\beta_1 + k_2\alpha_2\beta_2 + k_3\alpha_3\beta_3, \\ &\cdots\cdots\cdots\cdots\cdots\cdots \\ &\cdots\cdots\cdots\cdots\cdots\cdots \end{aligned}$$

The kinetic energy of the system is

$$T = \tfrac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2).$$

The equations of free vibrations of the particle then are

$$\left.\begin{aligned} m\ddot{x} + c_{11}x + c_{12}y + c_{13}z &= 0, \\ m\ddot{y} + c_{12}x + c_{22}y + c_{23}z &= 0, \\ m\ddot{z} + c_{13}x + c_{23}y + c_{33}z &= 0. \end{aligned}\right\} \tag{g}$$

Taking the solution of these equations in the form (b), we obtain

$$\left.\begin{aligned} (c_{11} - mp^2)\lambda_1 + c_{12}\lambda_2 + c_{13}\lambda_3 &= 0, \\ c_{12}\lambda_1 + (c_{22} - mp^2)\lambda_2 + c_{23}\lambda_3 &= 0, \\ c_{13}\lambda_1 + c_{23}\lambda_2 + (c_{33} - mp^2)\lambda_3 &= 0. \end{aligned}\right\} \tag{h}$$

These equations may give, for λ_1, λ_2, λ_3, solutions different from zero only if their determinant vanishes. Putting this determinant equal to zero, we obtain in this case a cubic equation in p^2 that gives three positive roots p_1^2, p_2^2, and p_3^2. Taking one of these roots, say p_1^2, and substituting it into Eqs. (h), we obtain the ratios λ_1'/λ_2' and λ_1'/λ_3', which define the mode of vibration having the frequency p_1. The corresponding motion is

$$x = \lambda_1'\sin(p_1 t + \alpha_1), \qquad y = \lambda_2'\sin(p_1 t + \alpha_1), \qquad z = \lambda_3'\sin(p_1 t + \alpha_1). \tag{i}$$

[1] The initial tensions in the spring balancing the weight of the particle will not enter in our consideration and can be disregarded.

We see that during vibration, the coordinates of the particle always are in the same ratio, which indicates that the particle oscillates along a certain straight line passing through the origin O of the coordinates. If α', β', γ' are the direction cosines of this line of motion and C' is the amplitude of the vibration, we have

$$\lambda_1' = \alpha' C', \qquad \lambda_2' = \beta' C', \qquad \lambda_3' = \gamma' C'$$

and

$$(C')^2 = (\lambda_1')^2 + (\lambda_2')^2 + (\lambda_3')^2.$$

As we have already seen, Eqs. (h) give only the frequencies of principal vibrations and the corresponding amplitude ratios. The absolute values of the amplitudes remain indefinite as long as we do not take into consideration the initial conditions of vibration. We can take

$$\lambda_1' = \alpha', \qquad \lambda_2' = \beta', \qquad \lambda_3' = \gamma'.$$

Then

$$(\lambda_1')^2 + (\lambda_2')^2 + (\lambda_3')^2 = 1, \tag{j}$$

and

$$x = \alpha' \sin(p_1 t + \alpha_1), \qquad y = \beta' \sin(p_1 t + \alpha_1), \qquad z = \gamma' \sin(p_1 t + \alpha_1).$$

Similarly, for the frequency p_2, we shall obtain

$$x = \alpha'' \sin(p_2 t + \alpha_2), \qquad y = \beta'' \sin(p_2 t + \alpha_2), \qquad z = \gamma'' \sin(p_2 t + \alpha_2).$$

Substituting the amplitudes (α', β', γ') and (α'', β'', γ'') of the two normal modes of vibration into Eqs. (h), we obtain the following two systems of equations:

$$\left.\begin{aligned} mp_1^2\alpha' &= c_{11}\alpha' + c_{12}\beta' + c_{13}\gamma', \\ mp_1^2\beta' &= c_{12}\alpha' + c_{22}\beta' + c_{23}\gamma', \\ mp_1^2\gamma' &= c_{13}\alpha' + c_{23}\beta' + c_{33}\gamma', \end{aligned}\right\} \tag{k}$$

and

$$\left.\begin{aligned} mp_2^2\alpha'' &= c_{11}\alpha'' + c_{12}\beta'' + c_{13}\gamma'', \\ mp_2^2\beta'' &= c_{12}\alpha'' + c_{22}\beta'' + c_{23}\gamma'', \\ mp_2^2\gamma'' &= c_{13}\alpha'' + c_{23}\beta'' + c_{33}\gamma''. \end{aligned}\right\} \tag{l}$$

If the first of Eqs. (k) is multiplied by α'', the second by β'', and the third by γ'' and then they are added together, we shall obtain, on the right-hand side, the same summation as we will get by multiplying, respectively, by α', β', γ' and adding together Eqs. (l). This proves that

$$mp_1^2(\alpha'\alpha'' + \beta'\beta'' + \gamma'\gamma'') = mp_2^2(\alpha'\alpha'' + \beta'\beta'' + \gamma'\gamma'');$$

and, since p_1^2 and p_2^2 are two different roots[1] of the frequency equation,

[1] We omit here the exceptional case where the frequency equation has two equal roots, which is of no practical importance.

we conclude that

$$\alpha'\alpha'' + \beta'\beta'' + \gamma'\gamma'' = 0. \tag{m}$$

This shows that in the two principal vibrations, the particle is moving along two perpendicular straight lines. Similar conclusions will be obtained if we consider principal vibrations with the frequencies p_1 and p_3 or p_2 and p_3, and we may conclude that the three principal modes of vibration of the particle in Fig. 225 are three harmonic motions along three mutually perpendicular straight lines. The direction cosines of these lines are determined from such equations as (k) and (l). If, from the very beginning, we had taken the coordinate axes in the directions of the principal vibrations, each equation of motion would have contained only one coordinate and these coordinates would be the principal coordinates of the system.

The above conclusion can be reached also in a different way. Giving to the particle m in Fig. 225 some displacement s from the position of equilibrium and then releasing it without initial velocity, we usually shall get a complicated motion of the particle consisting of a combination of the three principal modes of vibration. This results from the fact that for an arbitrary displacement of the particle, the resultant R of the forces in the three springs has a direction different from the direction of the displacement. But the motion is much simplified if we select the initial displacement of the particle in such a way that its direction coincides with the direction of the resultant force of the springs. If this condition is fulfilled, the released particle will move along the line coinciding with the initial displacement and we shall obtain a simple harmonic vibration along this line. To find the proper directions, assume that the above condition is fulfilled and that x, y, z are the three components of the displacement s. Then from expression (f), for the potential energy we find, for the three components of the resultant force R acting on the particle,

$$\left.\begin{aligned} X &= -\frac{\partial V}{\partial x} = -c_{11}x - c_{12}y - c_{13}z, \\ Y &= -\frac{\partial V}{\partial y} = -c_{12}x - c_{22}y - c_{23}z, \\ Z &= -\frac{\partial V}{\partial z} = -c_{13}x - c_{23}y - c_{33}z. \end{aligned}\right\} \tag{n}$$

Since the line of action of the resultant R coincides with the displacement s, we have

$$X = -R\frac{x}{s}, \qquad Y = -R\frac{y}{s}, \qquad Z = -R\frac{z}{s}.$$

Substituting into Eqs. (n), we obtain

$$
\left.\begin{aligned}
\left(c_{11} - \frac{R}{s}\right)x + c_{12}y + c_{13}z &= 0,\\
c_{12}x + \left(c_{22} - \frac{R}{s}\right)y + c_{23}z &= 0,\\
c_{13}x + c_{23}y + \left(c_{33} - \frac{R}{s}\right)z &= 0.
\end{aligned}\right\} \qquad (o)
$$

Comparing these equations with Eqs. (*h*), we conclude that the components x,y,z of the displacement s will be in the same ratio as the amplitudes λ_1, λ_2, λ_3 of a principal mode of vibration. Hence, the directions of the displacements, for which the line of action of the resultant R coincides with the displacement, are the directions of the principal coordinate axes that we found before.

Several additional examples of free vibrations of systems with more than two degrees of freedom will now be considered.

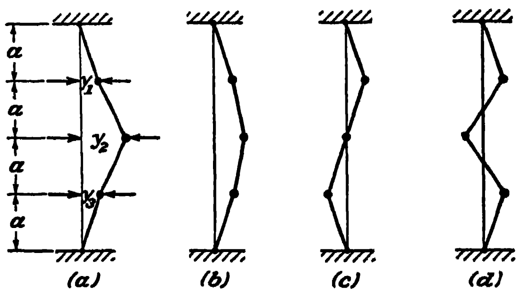

FIG. 226.

Example: The system shown in Fig. 226*a* consists of three equal masses m attached to the quarter points of a tightly stretched vertical string fixed at its ends. Neglecting the mass of the string and assuming that its initial tension S is very large and can be considered constant during lateral vibrations of the particles, find the fundamental frequencies and the corresponding modes of vibration.

Solution: The potential energy of the system is obtained by multiplying the tensile force S by the elongation of the string during vibration, which gives

$$V = \frac{1}{2a}S[y_1^2 + (y_2 - y_1)^2 + (y_3 - y_2)^2 + y_3^2] = \frac{S}{a}(y_1^2 + y_2^2 + y_3^2 - y_1y_2 - y_2y_3).$$

The kinetic energy of the system is

$$T = \frac{m}{2}(\dot{y}_1^2 + \dot{y}_2^2 + \dot{y}_3^2).$$

Substituting into Lagrangian equations (136), we obtain the equations of motion

$$
\begin{aligned}
m\ddot{y}_1 + \frac{S}{a}(2y_1 - y_2) &= 0,\\
m\ddot{y}_2 + \frac{S}{a}(2y_2 - y_1 - y_3) &= 0,\\
m\ddot{y}_3 + \frac{S}{a}(2y_3 - y_2) &= 0.
\end{aligned}
$$

Then assuming a particular solution

$$y_1 = \lambda_1 \sin(pt + \alpha), \qquad y_2 = \lambda_2 \sin(pt + \alpha), \qquad y_3 = \lambda_3 \sin(pt + \alpha)$$

and using the notation $\beta = S/ma$, we obtain

$$\begin{aligned} \lambda_1(p^2 - 2\beta) + \lambda_2\beta &= 0, \\ \lambda_1\beta + \lambda_2(p^2 - 2\beta) + \lambda_3\beta &= 0, \\ \lambda_2\beta + \lambda_3(p^2 - 2\beta) &= 0. \end{aligned}$$

Equating to zero the determinant of these equations, we get the following frequency equation:

$$(p^2 - 2\beta)(p^4 - 4p^2\beta + 2\beta^2) = 0,$$

the three roots of which are

$$p_1^2 = (2 - \sqrt{2})\beta, \qquad p_2^2 = 2\beta, \qquad p_3^2 = (2 + \sqrt{2})\beta.$$

The corresponding modes of vibration are shown in Fig. 226*b*,*c*,*d*.

Example: A vertical shaft with three identical and equidistant disks is fixed at the upper end and performs torsional vibrations (Fig. 227). Find the frequencies of the principal vibrations if the torsional rigidity of the shaft is C, the moment of inertia of each disk is I, and the length of the shaft is $3l$.

$$Ans. \quad p_1 = 0.445\sqrt{\frac{C}{lI}}, \; p_2 = 1.247\sqrt{\frac{C}{lI}}, \; p_3 = 1.801\sqrt{\frac{C}{lI}}.$$

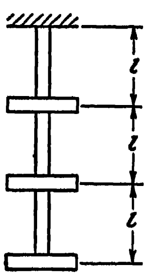

FIG. 227.

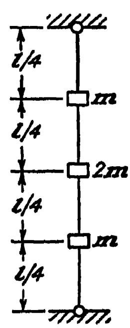

FIG. 228.

Example: A vertical prismatic bar with hinged ends carries three equidistant particles of masses m, $2m$, and m, as shown in Fig. 228. Find the principal frequencies of lateral vibrations of these particles in the plane of the figure, neglecting the mass of the bar.

Solution: To write equations of motion, we use here D'Alembert's principle. If lateral loads P_1, P_2, P_3 are applied to the particles, the lateral deflections will be

$$\begin{aligned} y_1 &= \frac{l^3}{768EI}(9P_1 + 11P_2 + 7P_3), \\ y_2 &= \frac{l^3}{768EI}(11P_1 + 16P_2 + 11P_3), \\ y_3 &= \frac{l^3}{768EI}(7P_1 + 11P_2 + 9P_3). \end{aligned}$$

When the bar is vibrating, we have to substitute the inertia forces $-m\ddot{y}_1$, $-2m\ddot{y}_2$,

$-m\ddot{y}_3$ for P_1, P_2, P_3. Assuming that

$$y_1 = \lambda_1 \sin(pt + \alpha'), \qquad y_2 = \lambda_2 \sin(pt + \alpha'), \qquad y_3 = \lambda_3 \sin(pt + \alpha')$$

and using the notation

$$\frac{l^3}{768EI} = \alpha,$$

we then obtain from the above equations of statics

$$\begin{aligned}
\lambda_1 &= \alpha m p^2 (9\lambda_1 + 22\lambda_2 + 7\lambda_3), \\
\lambda_2 &= \alpha m p^2 (11\lambda_1 + 32\lambda_2 + 11\lambda_3), \\
\lambda_3 &= \alpha m p^2 (7\lambda_1 + 22\lambda_2 + 9\lambda_3).
\end{aligned}$$

Letting $1/\alpha m p^2 = z$, we obtain the frequency equation

$$z^3 - 50z^2 + 124z - 56 = 0,$$

the roots of which are

$$z_1 = 47.41, \qquad z_2 = 2.00, \qquad z_3 = 0.59.$$

The corresponding frequencies are

$$p_1 = \sqrt{\frac{1}{\alpha m z_1}} = 4.026\sqrt{\frac{EI}{ml^3}}, \qquad p_2 = 19.60\sqrt{\frac{EI}{ml^3}}, \qquad p_3 = 37.3\sqrt{\frac{EI}{ml^3}}.$$

Example: Investigate vertical vibrations of the system shown in Fig. 229*a*, consisting of a rotor mounted on a flexible shaft that, in turn, is supported by a frame resting on a heavy foundation slab.

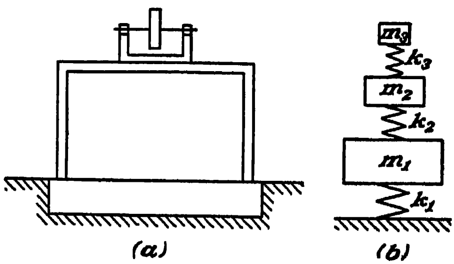

FIG. 229.

Solution: Taking into account flexibility of the shaft, flexibility of the frame, and also elasticity of the soil supporting the slab, we imagine an idealized system with three degrees of freedom as shown in Fig. 229*b*. From the known dimensions of the installation and from its elastic properties, the masses m_1, m_2, m_3 and the spring constants of the idealized system can be calculated in any particular case. For coordinates, we take the deflections x_1, x_2, x_3 of the springs from their equilibrium configurations. Then the kinetic and potential energies of the system are

$$\begin{aligned}
T &= \tfrac{1}{2}[m_1\dot{x}_1^2 + m_2(\dot{x}_1 + \dot{x}_2)^2 + m_3(\dot{x}_1 + \dot{x}_2 + \dot{x}_3)^2], \\
V &= \tfrac{1}{2}(k_1x_1^2 + k_2x_2^2 + k_3x_3^2),
\end{aligned}$$

and we obtain the following system of equations of motion:

$$\begin{aligned}
(m_1 + m_2 + m_3)\ddot{x}_1 + (m_2 + m_3)\ddot{x}_2 + m_3\ddot{x}_3 + k_1x_1 &= 0, \\
(m_2 + m_3)\ddot{x}_1 + (m_2 + m_3)\ddot{x}_2 + m_3\ddot{x}_3 + k_2x_2 &= 0, \\
m_3\ddot{x}_1 + m_3\ddot{x}_2 + m_3\ddot{x}_3 + k_3x_3 &= 0,
\end{aligned}$$

Proceeding as before, we obtain the frequency equation which, for known numerical values of the masses and spring constants, represents a cubic equation in p^2 with

numerical coefficients. When that equation is solved, the three principal modes of vibration will be obtained.

PROBLEMS

143. Set up the equations of motion, and derive the frequency equation for small lateral vibrations of the two masses m_1 and m_2 shown in Fig. 230. Assume that the tensile force S in the vertical string is large.

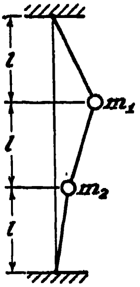

FIG. 230.

Ans. $p^2 = \dfrac{S(m_1 + m_2)}{m_1 m_2 l} \pm \sqrt{\dfrac{S^2(m_1 + m_2)^2}{m_1^2 m_2^2 l^2} - \dfrac{3S^2}{m_1 m_2 l^2}}.$

144. Three circular disks of moments of inertia I_1, I_2, I_3 are mounted on a horizontal shaft that can turn freely in its bearings as shown in Fig. 231. The torsional spring constants for the two portions

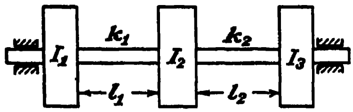

FIG. 231.

of the shaft are k_1 and k_2 as shown. Set up the equations of motion, and derive the frequency equation for torsional oscillations of the disks.

Ans. $(I_1I_2I_3)p^6 - [k_1(I_1I_3 + I_2I_3) + k_2(I_1I_2 + I_1I_3)]p^4 + k_1k_2(I_1 + I_2 + I_3)p^2 = 0.$

145. A bifilar pendulum of mass M and length a hangs in a vertical plane and has attached to it two identical single pendulums each of mass m and length b as shown in Fig. 232. Determine the roots of the frequency equation for small oscillations of the system in the plane of the figure.

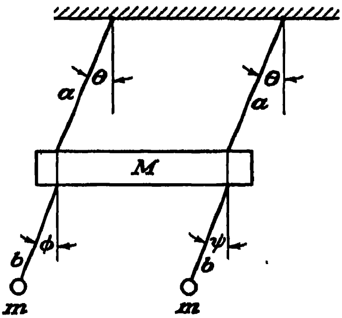

FIG. 232.

Ans. $p_1^2 = \dfrac{b}{g};\ p_{2,3}^2 = \dfrac{a+b}{2ab}\dfrac{M+2m}{M}g \pm \sqrt{\dfrac{(a+b)^2}{4a^2b^2}\left(\dfrac{M+2m}{M}\right)^2 g^2 - \dfrac{M+2m}{M}\dfrac{g_2}{ab}}.$

39. Approximate Methods of Calculating Principal Frequencies.—

We have seen in the preceding article that the calculation of frequencies of principal vibrations of a system requires the solution of the frequency equation that is obtained by equating to zero the determinant of a certain system of homogeneous linear equations. The evaluation of the coefficients of the frequency equation and the numerical solution of the equation itself become more and more involved as the number of degrees of freedom of the system increases. In the case of three degrees of freedom, we had to deal with a cubic equation. For systems with four degrees of freedom, we obtain a frequency equation of fourth degree, etc. In practical applications, we often have systems with many degrees of freedom, and the amount of work required to evaluate the coefficients and solve the frequency equation becomes prohibitive. In such cases, recourse must be made to some approximate method of calculating fre-

quencies without deriving the frequency equation.[1] In the following discussion we shall illustrate some of these methods in connection with the calculation of frequencies of lateral vibrations of a shaft with several masses (Fig. 233).

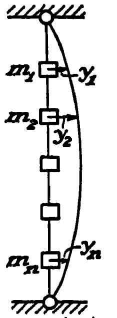

FIG. 233.

In many cases, we need only the lowest frequency p_1. This frequency can be calculated with good accuracy by using the Rayleigh method already discussed in Art. 20 (see page 170). In applying this method, we assume that the deflection curve of the shaft in the extreme positions of the lowest mode of vibration is of the same form as a statical deflection curve under the action of gravity forces m_1g, m_2g, . . . acting perpendicularly to the axis of the shaft. Denoting by λ_1, λ_2, . . . the statical deflections under the loads, we conclude that the strain energy of bending, when the vibrating shaft is in an extreme position, is

$$V = \frac{g}{2}(m_1\lambda_1 + m_2\lambda_2 + \cdots + m_n\lambda_n). \tag{a}$$

Then assuming the motions of the masses to be given by the expressions

$$y_1 = \lambda_1 \sin(p_1t + \alpha), \qquad y_2 = \lambda_2 \sin(p_1t + \alpha), \quad \ldots \quad y_n = \lambda_n \sin(p_1t + \alpha), \tag{b}$$

the kinetic energy of the system, in any configuration, is

$$T = \tfrac{1}{2}(m_1\dot{y}_1^2 + m_2\dot{y}_2^2 + \cdots + m_n\dot{y}_n^2).$$

Substituting expressions (b) for y_1, y_2, . . . y_n, we find that the maximum kinetic energy is

$$T_{max} = \tfrac{1}{2}p_1^2(m_1\lambda_1^2 + m_2\lambda_2^2 + \cdots + m_n\lambda_n^2). \tag{c}$$

This T_{max} occurs when the vibrating shaft passes through its unstrained middle position. Since we neglect dissipation of energy, expressions (c) and (a) must be equal and we obtain

$$p_1^2 = \frac{g(m_1\lambda_1 + m_2\lambda_2 + \cdots + m_n\lambda_n)}{m_1\lambda_1^2 + m_2\lambda_2^2 + \cdots + m_n\lambda_n^2}. \tag{151}$$

We see that the fundamental frequency p_1 is easily calculated if we know the statical deflections $\lambda_1, \lambda_2, \cdots \lambda_n$, which can be readily obtained by analytical or graphical methods.

[1] The various methods of solving this problem are extensively discussed in the book by F. van den Dungen, "Cours de technique des vibrations," Bruxelles, 1926. See also K. Hohenemser, "Die Methoden zur angenäherten Lösung von Eigenwertproblemen," Berlin, 1932.

For calculating the same frequency p_1, a method of successive approximations can be applied.[1] Considering again statical deflections under the action of lateral forces $m_1g, m_2g, \ldots m_ng$, we can write the expression for the deflection y_i at the point of application of the load m_ig in the following form:

$$y_i = g(\alpha_{i1}m_1 + \alpha_{i2}m_2 + \cdots + \alpha_{in}m_n), \tag{d}$$

where α_{ij} is the deflection at i produced by a unit load acting at j. The quantities α_{ij} are called *influence numbers*. All these numbers are obtained from one deflection curve produced by a unit load applied at point i. From Maxwell's reciprocal theorem,[2] it follows that $\alpha_{ij} = \alpha_{ji}$ and the ordinates of this curve at points 1, 2, . . . n give us $\alpha_{i1}, \alpha_{i2}, \ldots \alpha_{in}$, respectively. The statical equation (d) can be used also in the case of vibration of the shaft. We have only to substitute the inertia forces $-m_1\ddot{y}_1, -m_2\ddot{y}_2, \ldots -m_n\ddot{y}_n$ for gravity forces. This gives equations of motion in the form

$$y_i = -(\alpha_{i1}m_1\ddot{y}_1 + \alpha_{i2}m_2\ddot{y}_2 + \cdots + \alpha_{in}m_n\ddot{y}_n). \tag{e}$$

We shall have as many equations of this kind as the number of masses on the shaft in Fig. 233. Assuming that the shaft performs the fundamental mode of vibration and substituting expressions (b) for $y_1, y_2, \ldots y_n$, we obtain the following system of linear homogeneous equations:

$$\left.\begin{aligned} \lambda_1 &= p_1^2(\alpha_{11}m_1\lambda_1 + \alpha_{12}m_2\lambda_2 + \cdots + \alpha_{1n}m_n\lambda_n), \\ \lambda_2 &= p_1^2(\alpha_{21}m_1\lambda_1 + \alpha_{22}m_2\lambda_2 + \cdots + \alpha_{2n}m_n\lambda_n), \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots, \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots, \\ \lambda_n &= p_1^2(\alpha_{n1}m_1\lambda_1 + \alpha_{n2}m_2\lambda_2 + \cdots + \alpha_{nn}m_n\lambda_n). \end{aligned}\right\} \tag{f}$$

By equating the determinant of these equations to zero, we obtain the equation for a rigorous calculation of the principal frequencies. For an approximate calculation of these frequencies, we do not derive the frequency equation but work directly with Eqs. (f). We usually know approximately the deflection curve corresponding to the fundamental mode of vibration of the shaft and can select reasonably well the series $\lambda_{01}, \lambda_{02}, \ldots \lambda_{0n}$ of approximate values for the amplitudes $\lambda_1, \lambda_2, \ldots \lambda_n$. Substituting these values into the right-hand sides of Eqs. (f), we

[1] The method originated with L. Vianello, *Z. Ver. deut. Ing.*, vol. 42, p. 1436, 1898. It was used for calculating buckling loads for struts. Its application for calculating principal frequencies of shafts is discussed with many details in the book by C. B. Biezeno and R. Grammel, "Technische Dynamik," pp. 155–163, 829–836, Berlin, 1939.

[2] See the authors' "Theory of Structures," p. 250.

obtain first approximations $\lambda_{11}, \lambda_{12}, \ldots \lambda_{1n}$ for the amplitudes from equations of the form

$$\lambda_{1i} = p_1^2(\alpha_{i1}m_1\lambda_{01} + \alpha_{i2}m_2\lambda_{02} + \cdots + \alpha_{in}m_n\lambda_{0n}), \tag{g}$$

while the first approximation for p_1 will be of the form

$$p_1^2 \approx \frac{\lambda_{0i}}{\Sigma\alpha_{ij}m_j\lambda_{0j}}. \tag{h}$$

This first approximation for p_1^2, as obtained from each of Eqs. (f), in turn, will have somewhat varying values, but we can get a better approximation by averaging them. For this purpose, we multiply both numerator and denominator of Eq. (h) by λ_{1i} and use, as the first approximation, the value

$$p_1^2 = \frac{\Sigma\lambda_{0i}\lambda_{1i}}{\Sigma\lambda_{1i}\Sigma\alpha_{ij}m_j\lambda_{0j}}. \tag{152}$$

Calculations show that if the initial form of the curve is well chosen, this formula gives p_1^2 with an accuracy sufficient for practical application.

If a better approximation is required, we use, for the amplitudes in the right-hand sides of Eqs. (f), the first approximations $\lambda_{11}, \lambda_{12}, \ldots \lambda_{1n}$ from Eqs. (g). In this way, we obtain

$$\lambda_{2i} = p_1^2(\alpha_{i1}m_1\lambda_{11} + \alpha_{i2}m_2\lambda_{12} + \cdots + \alpha_{in}m_n\lambda_{1n}) \tag{i}$$

and

$$p_1^2 \approx \frac{\lambda_{1i}}{\Sigma\alpha_{ij}m_j\lambda_{1j}}.$$

Then averaging the values of p_1^2, as before, we find for the second approximation

$$p_1^2 \approx \frac{\Sigma\lambda_{1i}\lambda_{2i}}{\Sigma\lambda_{2i}\Sigma\alpha_{ij}m_j\lambda_{1j}}. \tag{153}$$

The calculation of further approximations can be continued in the same manner, and we can draw some conclusion about the convergence of the process by whether or not the values of p_1^2, as calculated from various equations of the system (f), are approaching the same limiting value. In this manner, the calculation of the fundamental frequency of vibration can be accomplished with any desired degree of accuracy.

Before we consider the calculation of higher frequencies of the shaft, let us prove a certain general property of the amplitudes $\lambda_1, \lambda_2, \ldots \lambda_n$ if they are related to two different principal modes of vibration. For this purpose, we shall use equations of motion in Lagrangian form instead of Eqs. (e) derived from D'Alembert's principle. Equation (d), giving the deflection at point i produced by any forces $P_1, P_2, \ldots P_n$ applied

at points 1, 2, . . . n, can be expressed in the form

$$y_i = \alpha_{i1}P_1 + \alpha_{i2}P_2 + \cdots + \alpha_{in}P_n.$$

Writing this equation for each of the points 1, 2, . . . n and solving for $P_1, P_2, \ldots P_n$, we obtain, for any force P_i,

$$P_i = c_{i1}y_1 + c_{i2}y_2 + \cdots + c_{in}y_n, \tag{j}$$

where $c_{i1}, c_{i2}, \ldots c_{in}$, are new constants. Equations (j) give the forces that must be applied to the shaft, at points 1, 2, . . . n, to produce deflections $y_1, y_2, \ldots y_n$. The forces that the shaft exerts on the masses $m_1, m_2, \ldots$ during vibration have opposite sign, and the equations of motion of these masses are

$$\begin{aligned}
m_1\ddot{y}_1 &= -(c_{11}y_1 + c_{12}y_2 + \cdots + c_{1n}y_n),\\
m_2\ddot{y}_2 &= -(c_{21}y_1 + c_{22}y_2 + \cdots + c_{2n}y_n),\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots,\\
m_n\ddot{y}_n &= -(c_{n1}y_1 + c_{n2}y_2 + \cdots + c_{nn}y_n).
\end{aligned}$$

Taking the solution of these equations in the form (b), we obtain

$$\left.\begin{aligned}
m_1p^2\lambda_1 &= c_{11}\lambda_1 + c_{12}\lambda_2 + \cdots + c_{1n}\lambda_n,\\
m_2p^2\lambda_2 &= c_{21}\lambda_1 + c_{22}\lambda_2 + \cdots + c_{2n}\lambda_n,\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots,\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots,\\
m_np^2\lambda_n &= c_{n1}\lambda_1 + c_{n2}\lambda_2 + \cdots + c_{nn}\lambda_n.
\end{aligned}\right\} \tag{k}$$

These equations have the same determinant as Eqs. (f) and will give the same frequency equation. Assume now that p_r^2 and p_s^2 are any two different roots of that equation and that $\lambda_{1r}, \lambda_{2r}, \ldots \lambda_{nr}; \lambda_{1s}, \lambda_{2s}, \ldots \lambda_{ns}$ are the amplitudes defining the corresponding modes of vibration. Then proceeding as in the case of a system with three degrees of freedom (see page 291), we obtain

$$\sum_{i=1}^{i=n} m_i\lambda_{ir}\lambda_{is} = 0. \tag{l}$$

This equation expresses the so-called *condition of orthogonality* of the principal modes of vibration.[1] Using Eq. (l), we can readily represent any shape of the vibrating shaft by superposition of the curves representing the principal modes of vibration. Let $\lambda_1', \lambda_2', \ldots \lambda_n'; \lambda_1'', \lambda_2'', \ldots \lambda_n''; \lambda_1''', \lambda_2''', \ldots \lambda_n'''$; etc., represent *sets of amplitudes* corresponding to the first, second, third, etc., modes of vibration. Take

[1] In the case of the system in Fig. 225, it states that the three principal modes of vibration take place along three mutually perpendicular straight lines as already noted on p. 291.

now any arbitrary deflection curve of the shaft defined by deflections $\lambda_{01}, \lambda_{02}, \ldots, \lambda_{0n}$ and assume that it is obtained by superposition of the curves corresponding to principal modes of vibration. Then

$$\left.\begin{aligned} \lambda_{01} &= c_1\lambda_1' + c_2\lambda_1'' + c_3\lambda_1''' + \cdots, \\ \lambda_{02} &= c_1\lambda_2' + c_2\lambda_2'' + c_3\lambda_2''' + \cdots, \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots, \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots, \\ \lambda_{0n} &= c_1\lambda_n' + c_2\lambda_n'' + c_3\lambda_n''' + \cdots. \end{aligned}\right\} \tag{m}$$

The constants $c_1, c_2, \ldots$ indicate what portions of the ordinates of the curves, representing principal modes of vibration, must be taken in order to form the arbitrarily taken curve with ordinates $\lambda_{01}, \lambda_{02}, \ldots \lambda_{0n}$. These constants will now be determined by using the *orthogonality condition* (*l*). To calculate, for example, the constant c_1, we multiply the first of Eqs. (*m*) by $m_1\lambda_1'$, the second by $m_2\lambda_2'$, etc. Now adding the equations and observing condition (*l*), we obtain

$$\sum_{i=1}^{i=n} \lambda_{0i}\lambda_i' m_i = c_1 \sum_{i=1}^{i=n} \lambda_i'^2 m_i.$$

Hence

$$c_1 = \frac{\sum_{i=1}^{i=n} \lambda_{0i}\lambda_i' m_i}{\sum_{i=1}^{i=n} \lambda_i'^2 m_i}. \tag{n}$$

In a similar manner, the other constants $c_2, c_3, \ldots$ can be calculated.

Returning now to the method of successive approximations used in solving Eqs. (*f*), it is important to note that the process of calculation described there converges and gives, at the limit, the lowest frequency p_1. If we wish to calculate in the same manner the second frequency p_2, we must select the initial amplitudes $\lambda_{01}, \lambda_{02}, \ldots \lambda_{0n}$ in such a way that the initial curve of the shaft will not only be suitable for representation of the second mode of vibration[1] but will not contain, as a component, the curve corresponding to the first mode of vibration. Only under this condition will the above-described method of successive approximations give us the desired second frequency p_2. To accomplish the proper selection of the initial curve with which the calculation should be started, we take first any curve suitable for representation of the second mode of vibration. Let $\lambda_{01}, \lambda_{02}, \ldots \lambda_{0n}$ be the ordinates of this curve. They can be considered as obtained by superposition of the curves represent-

[1] We usually know approximately how the second mode of vibration will look.

ing principal modes of vibration [see Eqs. (m)]: The first mode of vibration is already known from the previous calculation of p_1 (see page 298), and we have the set of values $\lambda_1', \lambda_2', \ldots \lambda_n'$ with good accuracy. Using these values, we calculate the constant c_1 by using Eq. (n). This constant defines in what proportion the assumed ordinates $\lambda_{01}, \lambda_{02}, \ldots \lambda_{0n}$ contain the ordinates of the fundamental mode of vibration. Cleaning the assumed curve of the portion belonging to the first mode of vibration, we obtain a new set of amplitudes

$$\bar{\lambda}_{01} = \lambda_{01} - c_1\lambda_1', \qquad \bar{\lambda}_{02} = \lambda_{02} - c_1\lambda_2', \; \ldots, \qquad \bar{\lambda}_{0n} = \lambda_{0n} - c_1\lambda_n', \tag{o}$$

with which the calculation of p_2 can be started. Substituting these values into Eqs. (g) and changing p_1^2 to p_2^2, we obtain, as first approximations for the amplitudes of the second mode of vibration,

$$\lambda_{1i} = p_2^2(\alpha_{i1}m_1\bar{\lambda}_{01} + \alpha_{i2}m_2\bar{\lambda}_{02} + \cdots + \alpha_{in}m_n\bar{\lambda}_{0n}).$$

Before going into a calculation of second approximations, we again clean the obtained set of amplitudes $\lambda_{11}, \lambda_{12}, \ldots \lambda_{1n}$ of the first mode of vibration as was done above [see Eqs. (o)] and use, as first approximations, the cleaned values $\bar{\lambda}_{11}, \bar{\lambda}_{12}, \ldots \bar{\lambda}_{1n}$. The second approximations then will be

$$\lambda_{2i} = p_2^2(\alpha_{i1}m_1\bar{\lambda}_{11} + \alpha_{i2}m_2\bar{\lambda}_{12} + \cdots + \alpha_{in}m_n\bar{\lambda}_{1n})$$

and approximately

$$p_2^2 \approx \frac{\bar{\lambda}_{1i}}{\Sigma\alpha_{ij}m_j\bar{\lambda}_{1j}}. \tag{p}$$

A better accuracy for the second frequency is obtained by averaging, as before, the values (p), which gives

$$p_2^2 = \frac{\Sigma\lambda_{2i}\bar{\lambda}_{1i}}{\Sigma\lambda_{2i}\Sigma\alpha_{ij}m_j\bar{\lambda}_{1j}}. \tag{154}$$

It is seen that in calculating the second frequency, we have, at each step, to clean the assumed curve of the component of the first mode of vibration. Only in this way can convergence of the process be realized. If we wish to calculate the third frequency by the same process of successive approximations, we have, before starting with calculations, to clean the assumed set of amplitudes $\lambda_{01}, \lambda_{02}, \ldots \lambda_{0n}$ of the components of the first and of the second modes of vibration, as explained above, and work with quantities $\bar{\lambda}_{01}, \bar{\lambda}_{02}, \ldots \bar{\lambda}_{0n}$.

It is seen that the amount of work in frequency calculations increases with the order of the frequency; and to save time, it is advantageous, beyond a certain order, to proceed not from the lowest but from the highest frequency of the system. For this purpose, instead of Eqs. (f), we

use Eqs. (k), which give

$$\left.\begin{aligned}
\lambda_1 &= \frac{1}{p^2}\left(\frac{c_{11}}{m_1}\lambda_1 + \frac{c_{12}}{m_1}\lambda_2 + \cdots + \frac{c_{1n}}{m_1}\lambda_n\right),\\
\lambda_2 &= \frac{1}{p^2}\left(\frac{c_{21}}{m_2}\lambda_1 + \frac{c_{22}}{m_2}\lambda_2 + \cdots + \frac{c_{2n}}{m_2}\lambda_n\right),\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots,\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots,\\
\lambda_n &= \frac{1}{p^2}\left(\frac{c_{n1}}{m_n}\lambda_1 + \frac{c_{n2}}{m_n}\lambda_2 + \cdots + \frac{c_{nn}}{m_n}\lambda_n\right).
\end{aligned}\right\} \qquad (q)$$

Instead of p^2, we have now the factor $1/p^2$ on the right-hand side of the equations, and the calculations that previously were used for the lowest mode of vibration will now be convergent for the highest mode of vibration, since for this mode the factor $1/p^2$ becomes the smallest.

To show the application of the method of successive approximations, let us consider the case of three masses as shown in Fig. 228, page 293. Equations (f) in this case are

$$\left.\begin{aligned}
\lambda_1 &= \alpha m p^2(9\lambda_1 + 22\lambda_2 + 7\lambda_3),\\
\lambda_2 &= \alpha m p^2(11\lambda_1 + 32\lambda_2 + 11\lambda_3),\\
\lambda_3 &= \alpha m p^2(7\lambda_1 + 22\lambda_2 + 9\lambda_3),
\end{aligned}\right\} \qquad (r)$$

where $\alpha = l^3/768EI$. As a rough approximation for the first mode of vibration, we take

$$\lambda_{01} = 1, \qquad \lambda_{02} = 2, \qquad \lambda_{03} = 1.$$

Substituting into the right-hand sides of Eqs. (r), we obtain

$$\lambda_{11} = 60\alpha m p^2, \qquad \lambda_{12} = 86\alpha m p^2, \qquad \lambda_{13} = 60\alpha m p^2. \qquad (s)$$

With these values again substituted in Eqs. (r), we find, as second approximations for the amplitudes,

$$\left.\begin{aligned}
\lambda_{21} &= (\alpha m p^2)^2(9\cdot 60 + 22\cdot 86 + 7\cdot 60),\\
\lambda_{22} &= (\alpha m p^2)^2(11\cdot 60 + 32\cdot 86 + 11\cdot 60),\\
\lambda_{23} &= (\alpha m p^2)^2(7\cdot 60 + 22\cdot 86 + 9\cdot 60).
\end{aligned}\right\} \qquad (t)$$

To obtain the frequency of the fundamental mode of vibration, we substitute the values (s) instead of $\lambda_{21}, \lambda_{22}, \lambda_{23}$ into Eqs. (t), which gives

$$\left.\begin{aligned}
p_1^2 &= \frac{60}{\alpha m(9\cdot 60 + 22\cdot 86 + 7\cdot 60)} = \frac{1}{47.53\alpha m},\\
p_1^2 &= \frac{86}{\alpha m(11\cdot 60 + 32\cdot 86 + 11\cdot 60)} = \frac{1}{47.35\alpha m},\\
p_1^2 &= \frac{60}{\alpha m(7\cdot 60 + 22\cdot 86 + 9\cdot 60)} = \frac{1}{47.53\alpha m}.
\end{aligned}\right\} \qquad (u)$$

These values are scattered very little, which indicates that the values (t) for the amplitudes are accurate enough. Averaging the values (u) by Eq. (153), we obtain

$$p_1^2 = \frac{1}{47.45\alpha m},$$

which differs from the solution of the frequency equation, given on page 294, only by about one-tenth of 1 per cent.

To get the second frequency by successive approximations, we observe that the second mode of vibration has an inflection point and assume

$$\lambda_{01} = 1, \qquad \lambda_{02} = 0, \qquad \lambda_{03} = -1.$$

Proceeding as explained before, we have to clean the assumed deflection curve of the component corresponding to the first mode of vibration. For this purpose, we use Eq. (n). Substituting the values (t) for λ_i' in this equation, we find that c_1 vanishes, which indicates that the assumed deflection curve does not contain a component corresponding to the fundamental mode of vibration. In such a case, we substitute the assumed values λ_{01}, λ_{02}, λ_{03} into the right-hand sides of Eqs. (r) and obtain the first approximations

$$\lambda_{11} = 2p^2\alpha m, \qquad \lambda_{12} = 0, \qquad \lambda_{13} = -2p^2\alpha m. \tag{v}$$

The amplitude ratios remain the same as we assumed initially and continue to remain unchanged as we calculate further approximations. This indicates that the initially assumed deflection curve is the true one for the second mode of vibration of the shaft. The corresponding frequency is obtained from the first of Eqs. (v) by substituting $\lambda_{01} = 1$ for λ_{11}, which gives

$$p_2^2 = \frac{1}{2\alpha m}.$$

This result coincides with the rigorous solution given on page 294.

To obtain the third frequency p_3, we observe that the third mode of vibration has two inflection points and assume

$$\lambda_{01} = 1, \qquad \lambda_{02} = -1, \qquad \lambda_{03} = 1.$$

Before substituting these values into Eqs. (r), we have to clean the assumed curve of the components of both the first and second modes of vibration. Using, for this purpose, Eq. (n) and taking the values $\lambda_1' = 1$, $\lambda_2' = 1.428$, and $\lambda_3' = 1$, which are in the same ratios as the values (t), we obtain

$$c_1 = -0.141.$$

This indicates that to clean the assumed curve of the component corresponding to the first mode of vibration, we have to subtract from

λ_{01}, λ_{02}, and λ_{03} the quantities $-1(0.141)$, $-1.428(0.141)$, $-1(0.141)$, respectively, which gives

$$\bar{\lambda}_{01} = 1.141, \qquad \bar{\lambda}_{02} = -0.800, \qquad \bar{\lambda}_{03} = 1.141.$$

To clean the assumed curve of the component corresponding to the second mode of vibration, we have to use, in Eq. (n), the amplitudes λ_i'' instead of λ_i', which are given by Eqs. (v). The summation in the numerator vanishes in this case, indicating that the assumed curve does not contain the component corresponding to the second mode of vibration. Hence, the quantities $\bar{\lambda}_{01}$, $\bar{\lambda}_{02}$, and $\bar{\lambda}_{03}$ can be used for calculating the frequency p_3. Substituting them into the right-hand sides of Eqs. (r), we obtain

$$\begin{aligned}
\lambda_{11} &= \alpha m p^2(9 \times 1.141 - 22 \times 0.800 + 7 \times 1.141) = 0.66\alpha m p^2,\\
\lambda_{12} &= \alpha m p^2(11 \times 1.141 - 32 \times 0.800 + 11 \times 1.141) = -0.50\alpha m p^2,\\
\lambda_{13} &= \alpha m p^2(7 \times 1.141 - 22 \times 0.800 + 9 \times 1.141) = 0.66\alpha m p^2.
\end{aligned}$$

Substituting $\bar{\lambda}_{01}$, $\bar{\lambda}_{02}$, $\bar{\lambda}_{03}$ for λ_{11}, λ_{12}, λ_{13}, we obtain approximate values for the third frequency as follows:

$$p_3^2 = \frac{1.141}{0.66\alpha m} = \frac{1.73}{\alpha m}, \qquad p_3^2 = \frac{0.800}{0.50\alpha m} = \frac{1.60}{\alpha m}, \qquad p_3^2 = \frac{1.73}{\alpha m}.$$

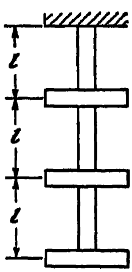

FIG. 234.

Averaging these values by Eq. (153), we obtain

$$p_3^2 = \frac{1.69}{\alpha m},$$

which coincides with the previously obtained value on page 294.

Example: Using the method of successive approximations, calculate the natural frequencies of torsional vibration of the system in Fig. 234. Each of the three disks has the same moment of inertia I, and the three portions of the shaft have the same tortional rigidity GJ.

Solution: Denoting by ϕ_1, ϕ_2, ϕ_3 the angles of rotation of the disks from their positions of equilibrium, we find the following expressions for kinetic and potential energy:

$$T = \frac{I}{2} \sum_{i=1}^{i=3} \dot{\phi}_i^2,$$

$$V = \frac{GJ}{2l} [\phi_1^2 + (\phi_2 - \phi_1)^2 + (\phi_3 - \phi_2)^2].$$

The equations of motion, then, are

$$\begin{aligned}
-I\ddot{\phi}_1 &= \frac{GJ}{l}(2\phi_1 - \phi_2),\\
-I\ddot{\phi}_2 &= \frac{GJ}{l}(-\phi_1 + 2\phi_2 - \phi_3),\\
-I\ddot{\phi}_3 &= \frac{GJ}{l}(-\phi_2 + \phi_3).
\end{aligned}$$

Substituting

$$\phi_1 = \lambda_1 \sin(pt - \alpha), \qquad \phi_2 = \lambda_2 \sin(pt - \alpha), \qquad \phi_3 = \lambda_3 \sin(pt - \alpha),$$

we obtain

$$\left.\begin{aligned} \left(2 - \frac{Ip^2l}{GJ}\right)\lambda_1 - \lambda_2 &= 0, \\ -\lambda_1 + \left(2 - \frac{Ip^2l}{GJ}\right)\lambda_2 - \lambda_3 &= 0, \\ -\lambda_2 + \left(1 - \frac{Ip^2l}{GJ}\right)\lambda_3 &= 0. \end{aligned}\right\} \tag{w}$$

The frequency equation is obtained from the determinant

$$\begin{vmatrix} 2 - \dfrac{Ip^2l}{GJ} & -1 & 0 \\ -1 & 2 - \dfrac{Ip^2l}{GJ} & -1 \\ 0 & -1 & 1 - \dfrac{Ip^2l}{GJ} \end{vmatrix} = 0.$$

This gives us a cubic equation from which

$$p_1 = 0.445\sqrt{\frac{GJ}{Il}}, \qquad p_2 = 1.247\sqrt{\frac{GJ}{Il}}, \qquad p_3 = 1.801\sqrt{\frac{GJ}{Il}}.$$

To get the frequencies by successive approximations, we write Eqs. (w) in the following form:

$$\left.\begin{aligned} \lambda_1 &= \frac{1}{\alpha p^2}(2\lambda_1 - \lambda_2), \\ \lambda_2 &= \frac{1}{\alpha p^2}(-\lambda_1 + 2\lambda_2 - \lambda_3), \\ \lambda_3 &= \frac{1}{\alpha p^2}(-\lambda_2 + \lambda_3), \end{aligned}\right\} \tag{x}$$

where $\alpha = Il/GJ$. These equations have the same form as Eqs. (q) and are suitable for calculating the highest frequency p_3. For this mode, we assume alternate signs for the angles of rotation and take

$$\lambda_{01} = 1, \qquad \lambda_{02} = -1, \qquad \lambda_{03} = 1.$$

Substituting in the right-hand sides of Eqs. (x), we obtain

$$\lambda_{11} = \frac{1}{\alpha p^2}\cdot 3, \qquad \lambda_{12} = \frac{1}{\alpha p^2}\cdot 0, \qquad \lambda_{13} = \frac{1}{\alpha p^2}\cdot 2.$$

Substituting again in Eqs. (x), we find

$$\lambda_{21} = \frac{1}{(\alpha p^2)^2}\cdot 6, \qquad \lambda_{22} = \frac{1}{(\alpha p^2)^2}(-5), \qquad \lambda_{23} = \frac{1}{(\alpha p^2)^2}\cdot 2.$$

The next calculation gives

$$\lambda_{31} = \frac{1}{(\alpha p^2)^3}\cdot 17, \qquad \lambda_{32} = \frac{1}{(\alpha p^2)^3}(-18), \qquad \lambda_{33} = \frac{1}{(\alpha p^2)^3}\cdot 7.$$

Substituting λ_{21}, λ_{22}, λ_{23} for λ_{31}, λ_{32}, λ_{33}, we find

$$p_3{}^2 = \frac{1}{\alpha}\frac{17}{6}, \qquad p_3{}^2 = \frac{1}{\alpha}\frac{18}{5}, \qquad p_3{}^2 = \frac{1}{\alpha}\frac{7}{2}.$$

Averaging these values by Eq. (153), we obtain

$$p_3^2 = \frac{1}{\alpha}\frac{17.17 + 18.18 + 7.7}{6.17 + 5.18 + 2.7} = \frac{1}{\alpha}3.21,$$

which gives

$$p_3 = \sqrt{3.21}\sqrt{\frac{GJ}{Il}} = 1.79\sqrt{\frac{GJ}{Il}}.$$

This result already is in good agreement with the rigorous solution given above.

To obtain the lowest frequency, we have to use equations of the form (*f*). For this purpose, the equations of motion can be obtained by using D'Alembert's principle. Considering the inertia torques $-I\ddot{\phi}_1$, $-I\ddot{\phi}_2$, $-I\ddot{\phi}_3$, applied to the disks, we obtain the following angles of rotation

$$\phi_1 = -\frac{Il}{GJ}(\ddot{\phi}_1 + \ddot{\phi}_2 + \ddot{\phi}_3),$$
$$\phi_2 = -\frac{Il}{GJ}(\ddot{\phi}_1 + 2\ddot{\phi}_2 + 2\ddot{\phi}_3),$$
$$\phi_3 = -\frac{Il}{GJ}(\ddot{\phi}_1 + 2\ddot{\phi}_2 + 3\ddot{\phi}_3),$$

and the equations to be solved are

$$\lambda_1 = \alpha p^2(\lambda_1 + \lambda_2 + \lambda_3),$$
$$\lambda_2 = \alpha p^2(\lambda_1 + 2\lambda_2 + 2\lambda_3),$$
$$\lambda_3 = \alpha p^2(\lambda_1 + 2\lambda_2 + 3\lambda_3).$$

Assuming, for the fundamental mode of vibration,

$$\lambda_{01} = 1, \qquad \lambda_{02} = 2, \qquad \lambda_{03} = 4,$$

successive approximations give

$$\lambda_{11} = \alpha p^2 \cdot 7, \qquad \lambda_{12} = \alpha p^2 \cdot 13, \qquad \lambda_{13} = \alpha p^2 \cdot 17,$$
$$\lambda_{21} = (\alpha p^2)^2 \cdot 37, \qquad \lambda_{22} = (\alpha p^2)^2 \cdot 67, \qquad \lambda_{23} = (\alpha p^2)^2 \cdot 84,$$
$$p_1^2 = \frac{7}{37\alpha}, \qquad p_1^2 = \frac{13}{67\alpha}, \qquad p_1^2 = \frac{17}{84\alpha}.$$

Averaging the p-squares by Eq. (153), we get

$$p_1 = \sqrt{\frac{1}{\alpha}\frac{7\cdot 37 + 13\cdot 67 + 17\cdot 84}{37^2 + 67^2 + 84^2}} = \sqrt{\frac{0.1981}{\alpha}} = 0.445\sqrt{\frac{GJ}{Il}},$$

which agrees with the above rigorous solution.

The calculation of p_2 is left to the student.

40. Systems with Infinite Number of Degrees of Freedom.—As an example of such systems, let us consider lateral vibrations of a prismatic bar with simply supported ends (Fig. 235). The deflection of the vibrating bar from its position of equilibrium can be represented by the trigonometric series (see page 204)

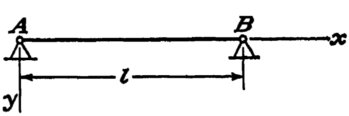

FIG. 235.

$$y = q_1 \sin\frac{\pi x}{l} + q_2 \sin\frac{2\pi x}{l} + q_3 \sin\frac{3\pi x}{l} + \cdots, \tag{a}$$

and the quantities $q_1, q_2, \ldots$, defining at any instant the shape of the vibrating bar, can be taken as the generalized coordinates in this case. The potential energy of bending, from page 168, is

$$V = \frac{EI}{2}\int_0^l \left(\frac{d^2y}{dx^2}\right)^2 dx = \frac{EI\pi^4}{4l^3}\sum_{i=1}^{\infty} i^4 q_i^2, \tag{b}$$

and the kinetic energy, during vibration, is

$$T = \frac{\gamma A}{2g}\int_0^l \dot{y}^2 dx,$$

where γA is the weight per unit length of the bar. Substituting for y its expression (a) and performing the indicated integration, we obtain

$$T = \frac{\gamma A l}{4g}\sum_{i=1}^{\infty} \dot{q}_i^2. \tag{c}$$

We see that the expressions for both kinetic and potential energy contain only squares of coordinates and corresponding velocities, respectively, which indicates that the selected coordinates are the *principal coordinates* of the system. Then for free vibrations, the Lagrangian equation (136) gives

$$\frac{\gamma A l}{2g}\ddot{q}_i + \frac{EI\pi^4 i^4}{2l^3} q_i = 0. \tag{d}$$

Introducing the notation

$$\frac{EIg}{A\gamma} = a^2, \tag{e}$$

we write the general solution of Eq. (d) in the form

$$q_i = A_i \cos\frac{i^2\pi^2 at}{l^2} + B_i \sin\frac{i^2\pi^2 at}{l^2}.$$

The corresponding principal vibrations are

$$y = \left(A_i \cos\frac{i^2\pi^2 at}{l^2} + B_i \sin\frac{i^2\pi^2 at}{l^2}\right)\sin\frac{i\pi x}{l}. \tag{f}$$

During any one of these principal modes of vibration, the bar is divided in i equal portions, the division points being the inflection points of the axis of the bar. The angular frequencies of these principal vibrations are

$$p_i = \frac{i^2\pi^2 a}{l^2} = \frac{i^2\pi^2}{l^2}\sqrt{\frac{EIg}{A\gamma}}, \tag{g}$$

and the corresponding periods are

$$\tau_i = \frac{2\pi}{p_i} = \frac{2l^2}{i^2\pi}\sqrt{\frac{A\gamma}{EIg}}. \tag{h}$$

The general case of vibration of the bar is obtained by superposition of the principal vibrations (*f*), which gives

$$y = \sum_{i=1}^{\infty}(A_i\cos p_it + B_i\sin p_it)\sin\frac{i\pi x}{l}. \tag{i}$$

The constants of integration A_i, B_i can be calculated in each particular case if the initial deflection curve of the bar and the initial velocity of each point of the bar are given. Assume, for example, that the initial deflection curve and the initial velocities are given by the equations

$$(y)_{t=0} = f(x), \qquad (\dot{y})_{t=0} = f_1(x).$$

Substituting for y its expression (*i*), we obtain

$$\sum_{i=1}^{\infty} A_i\sin\frac{i\pi x}{l} = f(x),$$

$$\sum_{i=1}^{\infty}\frac{i^2\pi^2a}{l^2}B_i\sin\frac{i\pi x}{l} = f_1(x).$$

The coefficients A_i and B_i are now obtained by multiplying these equations by $\sin(i\pi x/l)dx$ and integrating both sides from $x = 0$ to $x = l$. This gives

$$\left.\begin{aligned} A_i &= \frac{2}{l}\int_0^l f(x)\sin\frac{i\pi x}{l}dx, \\ B_i &= \frac{2l}{i^2\pi^2a}\int_0^l f_1(x)\sin\frac{i\pi x}{l}dx. \end{aligned}\right\} \tag{j}$$

Assume, for example, that the bar initially is in equilibrium and that by a blow a uniformly distributed vertical velocity v is communicated to the bar.[1] In such case, $f(x) = 0$, $f_1(x) = v$. Substituting into Eqs. (*j*), we obtain

$$A_i = 0, \qquad B_i = \frac{2l^2v}{i^3\pi^3a}\left|\cos\frac{i\pi x}{l}\right|_l^0,$$

and expression (*i*) gives

$$y = \frac{4l^2v}{\pi^3a}\left(\sin\frac{\pi x}{l}\sin p_1t + \frac{1}{3^3}\sin\frac{3\pi x}{l}\sin p_3t + \cdots\right).$$

[1] This condition is obtained also if the bar has a uniform vertical velocity v and then, at the instant $t = 0$, its ends are suddenly stopped.

It is seen that in this case, only the principal vibrations with uneven numbers of half waves will be produced, and the amplitudes of these vibrations rapidly diminish with increasing order i of the vibration.

In the case of forced vibration, there will be, in addition to elastic forces, some external forces depending on time. If Q_i is the external force corresponding to the coordinate q_i, the equation of motion is

$$\frac{\gamma A l}{2g}\ddot{q}_i + \frac{EI\pi^4 i^4}{2l^3} q_i = Q_i$$

and its general solution will be [see Eq. (31), page 50]

$$q_i = A_i \cos p_i t + B_i \sin p_i t + \frac{1}{p_i}\frac{2g}{A\gamma l}\int_0^t Q_i \sin p_i(t - t')dt'. \qquad (k)$$

The first two terms on the right-hand side represent free vibrations depending on initial conditions, while the last represents the vibrations produced by the force Q_i.

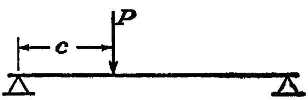

FIG. 236.

Consider, for example, vibrations produced in a bar by a pulsating force $P = P_0 \sin \omega t'$ applied at a distance c from the left end (Fig. 236). To find the generalized force corresponding to the coordinate q_i, we give to this coordinate an increment δq_i. The corresponding deflection of the bar is $\delta q_i \sin(i\pi x/l)$, and the work produced on this displacement by the force P is

$$P\,\delta q_i \sin\frac{i\pi c}{l}.$$

Hence

$$Q_i = P\sin\frac{i\pi c}{l} = P_0 \sin\omega t' \sin\frac{i\pi c}{l}.$$

Substituting into Eq. (k) and considering only the last term on the right-hand side, we obtain

$$\begin{aligned} q_i &= \frac{1}{p_i}\frac{2g}{A\gamma l}P_0 \sin\frac{i\pi c}{l}\int_0^t \sin\omega t' \sin p_i(t - t')dt' \\ &= \frac{2g}{A\gamma l}P_0 \sin\frac{i\pi c}{l}\left[\frac{1}{p_i^2 - \omega^2}\sin\omega t - \frac{\omega}{p_i(p_i^2 - \omega^2)}\sin p_i t\right] \qquad (l) \end{aligned}$$

It is seen that by applying the pulsating force P, we produce not only forced vibrations having the same frequency as the disturbing force but also free vibrations. In practical applications, various kinds of resistance always are present due to which free vibrations will be gradually damped out and only the forced vibrations finally remain. This latter vibration is also affected by friction; but in the case of small friction and if we are not very close to a resonance condition, this influence can be

neglected and we can put

$$q_i = \frac{2gP_0}{A\gamma l(p_i^2 - \omega^2)} \sin \frac{i\pi c}{l} \sin \omega t.$$

Substituting in expression (a), we obtain forced vibrations of the bar as follows:

$$y = \frac{2gP_0}{A\gamma l} \sum_{i=1}^{\infty} \frac{1}{p_i^2 - \omega^2} \sin \frac{i\pi c}{l} \sin \frac{i\pi x}{l} \sin \omega t. \tag{m}$$

It is seen that the pulsating force P produces a complicated vibration obtained by superposition of principal vibrations of various modes. The modes for which $\sin(i\pi c/l) = 0$ vanish from the summation (m). This indicates that the force P does not produce those vibrations for which the point of application of the force coincides with an inflection point. Each time when ω is approaching p_i, the corresponding term in the series (m) becomes predominate and we are approaching a condition of resonance. Since we have an infinite number of natural frequencies p_i, there will be an infinite number of *critical frequencies* ω of the pulsating force. When the frequency ω of the pulsating force is small in comparison with the fundamental frequency p_1 of the bar, we can neglect ω^2 in comparison with p_i^2 in expression (m) and obtain

$$y = \frac{2gP_0}{A\gamma l} \sum_{i=1}^{\infty} \frac{1}{p_i^2} \sin \frac{i\pi c}{l} \sin \frac{i\pi x}{l} \sin \omega t = \frac{2Pl^3}{EI\pi^4} \sum_{i=1}^{\infty} \frac{1}{i^4} \sin \frac{i\pi c}{l} \sin \frac{i\pi x}{l},$$

which represents the statical deflection of the bar (see page 205).

If there are several pulsating forces acting on the bar, the resulting vibration will be obtained by superimposing the vibrations produced by the individual forces.

The case of a continuously distributed pulsating force can be solved in the same manner; the summation only has to be replaced by integration along the length of the beam. Assume, for example, that the beam carries a uniformly distributed pulsating load[1] of intensity $w = w_0 \sin \omega t$. Substituting $w_0 dc$ for P_0 in expression (m), we obtain vibrations produced by one element of the pulsating load. Then integrating this expression with respect to c within the limits $c = 0$ and $c = l$, we obtain the required vibration

$$y = \frac{4gw_0}{A\gamma\pi} \sum_{i=1,3,5,\ldots}^{\infty} \frac{1}{i(p_i^2 - \omega^2)} \sin \frac{i\pi x}{l} \sin \omega t. \tag{n}$$

[1] Such case we have, for example, in analyzing vibrations of the side rod of a locomotive.

We see that in this case, only vibrations with odd numbers of half waves are produced. If ω is smaller than p_1, the series (n) converges rapidly and the first term alone gives the deflection curve with good accuracy. The maximum deflection at the middle then is

$$y_{max} = \frac{4gw_0}{A\gamma\pi}\frac{1}{p_1^2 - \omega^2} = \frac{4gw_0 l^4}{A\gamma\pi^5 a^2}\frac{1}{1 - \dfrac{\omega^2}{p_1^2}} \approx \frac{\delta_{st}}{1 - \dfrac{\omega^2}{p_1^2}}. \tag{o}$$

Again, as in the case of a system with one degree of freedom, the amplitude of vibration is obtained by multiplying the statical deflection of the beam by the magnification factor.

If a constant vertical force P is traveling with constant velocity v along the beam, we count time from the instant when the force enters the span and take $c = vt'$ (Fig. 236). The generalized force in such a case is[1]

$$Q_i = P \sin \frac{i\pi vt'}{l} = P \sin \omega_i t'.$$

Substituting this into the last term of Eq. (k), we obtain q_i, and the vibrations produced by the traveling force can be expressed in the form

$$y = \frac{2gP}{A\gamma l}\sum_{i=1}^{\infty}\left(\frac{\sin \omega_i t - \dfrac{\omega_i}{p_i}\sin p_i t}{p_i^2 - \omega_i^2}\right)\sin\frac{i\pi x}{l}. \tag{p}$$

If the time $l/v = \pi/\omega_1$ occupied by the force in crossing the beam is several times larger than the half-period $\tau_1/2 = \pi/p_1$ of the fundamental mode of vibration, we can neglect all except the first terms under the summation sign in expression (p) and write

$$y \approx \frac{2gP}{A\gamma l}\left(\frac{\sin \omega_1 t - \dfrac{\omega_1}{p_1}\sin p_1 t}{p_1^2 - \omega_1^2}\right)\sin\frac{\pi x}{l}.$$

Then, assuming that, in the most unfavorable case, the amplitudes of forced and free vibrations add together, we obtain

$$y_{max} = \frac{2gP}{A\gamma l}\left(\frac{1}{p_1^2 - \omega_1^2}\right)\left(1 + \frac{\omega_1}{p_1}\right).$$

Using the notation

$$\frac{vl}{a\pi} = \frac{\omega_1}{p_1} = \alpha,$$

we represent this maximum deflection in the form

$$y_{max} = \frac{2gP}{A\gamma l p_1^2}\frac{1+\alpha}{1-\alpha^2} \approx \frac{\delta_{st}}{1-\alpha}. \tag{q}$$

[1] We introduce the notation $\omega_i = i\pi v/l$ for simplicity of writing.

This formula can be used for an approximate calculation of the maximum deflection of a beam or bridge under the action of a moving load.[1]

In a similar manner vibrations of a beam produced by a pulsating traveling load can be investigated. Such an effect is produced on a bridge by the counterweights of a locomotive.

PROBLEMS

146. Investigate free vibrations of a highly stretched uniform string (Fig. 237). assuming that the tensile force S does not change during vibration.

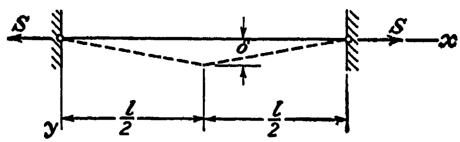

FIG. 237.

HINT: Use for deflections the series

$$y = \Sigma q_i \sin(i\pi x/l),$$

and for potential energy the expression

$$V = \frac{S}{2}\int_0^l \left(\frac{dy}{dx}\right)^2 dx$$

147. Find free vibrations of the string in the preceding problem if initially the middle of the string is displaced vertically by an amount δ and then released without initial velocity.

148. Calculate the natural frequencies of a prismatic bar simply supported at the ends if its maximum deflection under the action of its own weight is δ.

Ans. $p_i = \pi^2 i^2 \sqrt{5g/384\delta}.$

149. The deflection at the middle of a prismatic beam under its own weight is 1 in. Find the maximum deflection produced by a load equal to the weight of the beam and passing over it with uniform speed in 1 sec.

Ans. $\delta_{max} = \dfrac{8}{5(1-\alpha)}$ in., where $\alpha = \dfrac{\pi}{p_1} = \dfrac{1}{\pi}\sqrt{\dfrac{384}{5g}}.$

150. Prove that in the case of geometrically similar bridges, the periods of their natural vibrations are in the same ratio as their spans if the material is the same.

41. Vibrations about a Steady State of Motion.—Lagrangian equations used in the analysis of vibrations about a position of stable equilibrium can be applied also in studying vibrations of a system about a steady state of motion. We begin our discussion with an example having considerable practical importance. A circular disk rotates about its vertical geometric axis through O as shown in Fig. 238. At point A on the disk, distant r from the axis of rotation, a pendulum of length l and mass m is attached as shown. We assume that the pendulum is

[1] See paper by S. Timoshenko, *Bull. Polytech. Inst. Kiev*, 1908. See also, "Vibration Problems in Engineering," 2d ed., p. 358, Van Nostrand.

supported by a smooth horizontal surface so that motion of the system is confined to the horizontal plane of the disk; thus gravity forces and friction do not need to be considered. Assume first a steady motion of the system in which the pendulum takes a radial direction while the disk rotates with constant angular velocity ω. Naturally this steady motion is stable. Now, if by some slight disturbance, a small displacement of the pendulum from its radial position is produced, small vibrations of the pendulum and small fluctuations in the angular velocity of the disk will ensue. To study these vibrations, Lagrange's equations will be used. We denote by ϕ the small angle of inclination of the oscillating pendulum to the radius OA and by $\dot{\theta}$ the small fluctuations in the angular velocity of the disk. Then the kinetic energy of the disk alone is

$$\tfrac{1}{2}I(\omega + \dot{\theta})^2.$$

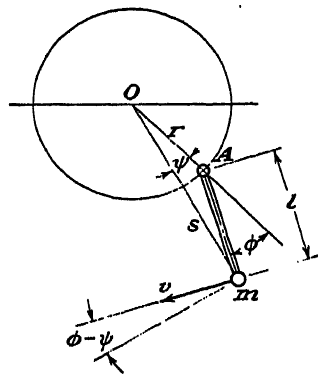

FIG. 238.

The total velocity v of the particle m is obtained as the geometric sum of the velocity $s(\omega + \dot{\theta})$, which it has in its motion together with the disk, and its velocity $l\dot{\phi}$ with respect to the disk. Then, as may be seen from Fig. 238,

$$v^2 = l^2\dot{\phi}^2 + s^2(\omega + \dot{\theta})^2 + 2l\dot{\phi}s(\omega + \dot{\theta})\cos(\phi - \psi). \tag{a}$$

For a small angle ϕ, we can assume

$$\psi = \frac{\phi l}{l + r}, \qquad \phi - \psi = \frac{\phi r}{l + r}.$$

We have also

$$s^2 \approx (l + r)^2 - lr\phi^2, \qquad s \approx (l + r)\left[1 - \frac{1}{2}\phi^2 \frac{lr}{(l + r)^2}\right].$$

Substituting into expression (a) and then making the expression for kinetic energy of the system, we obtain

$$T = \frac{1}{2}I(\omega + \dot{\theta})^2 + \frac{m}{2}\left\{l^2\dot{\phi}^2 + [(l + r)^2 - lr\phi^2](\omega + \dot{\theta})^2 \right.$$
$$\left. + 2l(l + r)\left[1 - \frac{1}{2}\phi^2 \frac{lr}{(l + r)^2}\right]\dot{\phi}(\omega + \dot{\theta})\left[1 - \frac{1}{2}\frac{\phi^2 r^2}{(l + r)^2}\right]\right\}. \tag{b}$$

Using this expression in Lagrangian equations and omitting, after differentiation, all terms containing powers or products of the small quantities ϕ, θ, $\dot{\phi}$, $\dot{\theta}$, we obtain the following system of two linear equations with constant coefficients:

$$\left.\begin{aligned}[I + m(l + r)^2]\ddot{\theta} + ml(l + r)\ddot{\phi} = 0,\\ \frac{l + r}{l}\ddot{\theta} + \ddot{\phi} + \frac{r\omega^2}{l}\phi = 0.\end{aligned}\right\} \qquad (c)$$

Eliminating $\ddot{\theta}$ from these equations and using the notation

$$I_0 = I + m(l + r)^2,$$

we obtain

$$\frac{I}{I_0}\ddot{\phi} + \frac{r\omega^2}{l}\phi = 0.$$

This shows that the rotating pendulum oscillates harmonically with the frequency

$$p = \omega\sqrt{\frac{r}{l}\frac{I_0}{I}}, \qquad (d)$$

which is proportional to the angular velocity of the steady rotation of the disk. This fact is of considerable practical interest as will be shown in the next article.

After this example, we now proceed to a consideration of the general case of small vibrations of a system about a steady state of motion. Defining the configuration of the system by coordinates $q_1, q_2, \ldots q_n$, we assume that the steady state of motion is defined by the equations

$$q_1 = f_1(t), \qquad q_2 = f_2(t), \qquad \cdots \qquad q_n = f_n(t). \qquad (e)$$

Now let us give to these coordinates, defining the steady state, some small disturbances $s_1, s_2, \ldots s_n$, so that

$$q_1 = f_1(t) + s_1, \qquad q_2 = f_2(t) + s_2, \qquad \cdots \qquad q_n = f_n(t) + s_n. \qquad (f)$$

In the case of the example discussed above, we had $f_1(t) = \omega t$, $f_2(t) = 0$, $s_1 = \theta$, and $s_2 = \phi$, so that expressions (f) were

$$q_1 = \omega t + \theta, \qquad q_2 = \phi.$$

Since, in general, $f_1(t), f_2(t), \ldots$ are known to us, the motion of the system after disturbance is completely defined by the quantities $s_1, s_2, \ldots$ and we can take these quantities as new coordinates. Then, in the case of forces having potential, the Lagrangian equations are

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{s}_i} - \frac{\partial T}{\partial s_i} = -\frac{\partial V}{\partial s_i}, \qquad (155)$$

in which T and V are the kinetic and potential energies calculated by using expressions (f). In calculating these energy functions, we treat $s_1, s_2, \ldots \dot{s}_1, \dot{s}_2, \ldots$ as small quantities and omit all terms containing

them in higher order than the second. Under such conditions, Eqs. (155) will become linear second-order equations; and if the initial motion was a steady motion, they will have constant coefficients[1] like Eqs. (c) above. Taking $s_1, s_2, \ldots \dot{s}_1, \dot{s}_2, \ldots$ equal to zero in these equations, we obtain the equations for steady motion. Then eliminating the corresponding terms from Eqs. (155), we obtain equations defining motion of the disturbed system with respect to the steady motion.

We assumed in the above discussion that the forces have potential. If, in addition to such forces, there are viscous friction forces, the additional terms, proportional to $\dot{s}_1, \dot{s}_2, \ldots$, will appear on the right-hand sides of Eqs. (155) but the equations will remain second-order differential equations with constant coefficients similar to those previously obtained for free vibration about a configuration of equilibrium. In the solution of these equations, we proceed in the usual way and assume

$$s_1 = C_1e^{rt}, \qquad s_2 = C_2e^{rt}, \qquad \cdots \qquad s_n = C_ne^{rt}. \tag{g}$$

Substituting into the differential equations, we obtain a system of homogeneous linear equations which may yield, for $C_1, C_2, \ldots$, solutions different from zero only if their determinant vanishes. In this way, we obtain for r an algebraic equation similar to a frequency equation and called the *characteristic equation*. The character of the motion of the system will depend on the roots of this equation. If all of its roots are real and negative, we conclude from expressions (g) that the assumed disturbances diminish with time and the initial steady motion is *stable*. If there are positive real roots, the assumed disturbances (g) are increasing with time and the initial steady motion is *unstable*. In the case of complex roots, $r = n \pm pi$, we can combine the corresponding solutions (g) and represent them in the form

$$Ce^{nt}\sin(pt + \alpha). \tag{h}$$

It is seen that the character of motion in this case depends on the sign of the real part n of the roots. If n is negative, expression (h) represents oscillations about the steady motion with gradually decreasing amplitude, which indicates that the steady motion is stable. If n is positive, the assumed disturbances are increasing with time and the initial steady motion is unstable. There are definite rules by which we can decide about the roots of the characteristic equation without solving it. For example, in the case of a cubic characteristic equation

$$a_0r^3 + a_1r^2 + a_2r + a_3 = 0, \tag{i}$$

[1] This is the definition of steady motion. The physical significance of this is that in the case of steady motion, a given disturbance always produces the same deviation from the initial motion independently of the instant at which that disturbance is produced.

the roots will be negative or will have a negative real part if all the coefficients are positive and if

$$a_1a_2 - a_0a_3 > 0. \qquad (j)$$

These conditions must be satisfied for stability of the corresponding steady motion.

In the case of a characteristic equation of the fourth degree

$$a_0r^4 + a_1r^3 + a_2r^2 + a_3r + a_4 = 0, \qquad (k)$$

it is necessary, for stability of steady motion, to have all the coefficients positive and, in addition, to have

$$a_3(a_1a_2 - a_0a_3) - a_1^2a_4 > 0. \qquad (l)$$

We shall now apply the general considerations regarding stability of a steady motion to the particular problem of oscillation of a steam-engine governor (Fig. 239). During rotation of the system about its vertical axis, the centrifugal forces of the flyballs produce compression of the spring and vertical motion of the sleeve B connected with the steam-supply throttle valve. If, for some reason, the speed of the engine increases, the flyballs rise higher and thereby lift the sleeve so that the opening of the steam valve C is reduced and the engine is throttled down. On the other hand, any downward movement of the flyballs increases the opening of the valve and consequently the amount of steam admitted to the engine. We denote by $\frac{1}{2}m_2$ the mass of each flyball considered as a particle and by m_1 the mass of the sleeve. The masses of the inclined bars and of the spring are neglected. As generalized coordinates, we take the angle of rotation ϕ of the governor about its vertical axis and the angle θ between the bars of the governor and the vertical axis as shown in the figure. The velocity of the center of each flyball has two components: (1) the velocity of rotation $\dot{\phi}(a + l \sin \theta)$ and (2) the velocity of lateral motion $l\dot{\theta}$. The vertical displacement of the sleeve from its lowest position is $2l(1 - \cos \theta)$, and its vertical velocity is $2l\dot{\theta} \sin \theta$. The kinetic energy of the system is

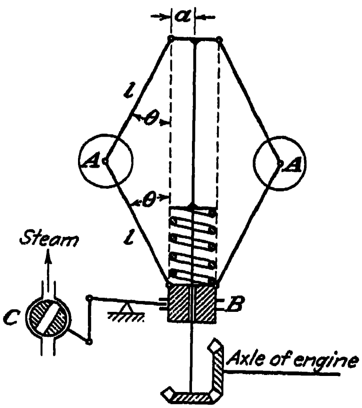

FIG. 239.

$$T = \tfrac{1}{2}m_2[(a + l \sin \theta)^2\dot{\phi}^2 + l^2\dot{\theta}^2] + \tfrac{1}{2}m_1 4l^2\sin^2\theta\dot{\theta}^2 + \tfrac{1}{2}I\dot{\phi}^2, \qquad (m)$$

where I is the *reduced moment of inertia* of the engine (see page 175).

The potential energy of the system consists of two parts: (1) the energy due to gravity forces

$$m_1g2l(1 - \cos\theta) + m_2gl(1 - \cos\theta)$$

and (2) the strain energy of the spring

$$\frac{4kl^2(1 - \cos\theta)^2}{2},$$

where k is the spring constant.[1] Then

$$V = gl(1 - \cos\theta)(2m_1 + m_2) + 2kl^2(1 - \cos\theta)^2. \tag{n}$$

We assume also that there is viscous friction in the sleeve and that the dissipation function (see page 279) is

$$F = \tfrac{1}{2}c(2l\dot\theta\sin\theta)^2. \tag{o}$$

Substituting expressions (m), (n), and (o) into Lagrangian equations (150), page 281, we obtain

$$\left.\begin{aligned} l^2(m_2 + 4m_1\sin^2\theta)\ddot\theta - m_2l\cos\theta(a + l\sin\theta)\dot\phi^2 - 4m_1l^2\sin\theta\cos\theta\dot\theta^2 \\ = -gl\sin\theta(2m_1 + m_2) - 4kl^2\sin\theta(1 - \cos\theta) - 4cl^2\sin^2\theta\dot\theta, \\ [I + m_2(a + l\sin\theta)^2]\ddot\phi = M, \end{aligned}\right\} \tag{p}$$

where M is the reduced torque acting on the engine shaft.

In the solution of Eqs. (p), let us begin with steady motion of the engine and substitute

$$\dot\phi = \omega, \qquad \ddot\phi = 0, \qquad \theta = \alpha, \qquad \dot\theta = \ddot\theta = 0, \qquad M = 0.$$

Then the first equation gives

$$m_2l\cos\alpha(a + l\sin\alpha)\omega^2 = gl\sin\alpha(2m_1 + m_2) + 4kl^2\sin\alpha(1 - \cos\alpha), \tag{q}$$

which defines the configuration of the system in its steady motion.

To consider small oscillations about the steady motion, we assume

$$\dot\phi = \omega + \dot\psi, \qquad \theta = \alpha + \eta,$$

where $\dot\psi$ denotes a small fluctuation in the angular velocity of rotation and η a small fluctuation in the angle of inclination θ. Then

$$\dot\phi^2 \approx \omega^2 + 2\omega\dot\psi, \qquad \sin\theta \approx \sin\alpha + \eta\cos\alpha,$$
$$\cos(\alpha + \eta) = \cos\alpha - \eta\sin\alpha.$$

Substituting into Eqs. (p) and using Eq. (q), we obtain, after omitting all terms containing squares or products of small quantities, the following system of equations:

[1] It is assumed that the spring force vanishes when $\theta = 0$.

$$\left.\begin{aligned} I_1\ddot{\eta} + b\dot{\eta} + d\eta - e\dot{\psi} &= 0, \\ I_0\ddot{\psi} &= -f\eta, \end{aligned}\right\} \tag{r}$$

in which $-f\eta$ denotes the decrease in torque acting on the shaft of the engine due to angular change η and the other symbols have the following meanings:

$$\begin{aligned} I_1 &= l^2(m_2 + 4m_1\sin^2\alpha), \\ I_0 &= I + m_2(a + l\sin\alpha)^2, \\ b &= 4cl^2\sin^2\alpha, \\ d &= m_2\omega^2[l\sin\alpha(a + l\sin\alpha) - l^2\cos^2\alpha] + gl\cos\alpha(2m_1 + m_2) \\ &\qquad + 4kl^2(\cos\alpha - \cos^2\alpha + \sin^2\alpha), \\ e &= 2\omega l m_2(a + l\sin\alpha). \end{aligned}$$

The oscillations of the governor with respect to its steady motion are defined by Eqs. (r). Substituting $\eta = C_1e^{rt}$ and $\psi = C_2e^{rt}$ into these equations, we obtain

$$\begin{aligned} C_1(I_1r^2 + br + d) - eC_2 &= 0, \\ C_1f + I_0rC_2 &= 0, \end{aligned}$$

and the corresponding characteristic equation is

$$I_0r(I_1r^2 + br + d) + ef = 0.$$

This is a cubic equation with all coefficients positive. The condition of stability (j) of the steady motion is

$$bdI_0^2 - efI_0I_1 > 0.$$

This condition imposes on the quantity b, depending on viscous friction, the following requirement:

$$b > \frac{efI_1}{dI_0}.$$

If this condition is not satisfied, the assumed steady motion of the governor is unstable. A sudden change in load on the engine will produce oscillations of the governor that will not be damped out gradually, and the known phenomenon of *hunting* of the governor will occur.

In the case where the engine is rigidly coupled to an electric generator, an additional term proportional to ψ in the second of Eqs. (r) will appear and the stability investigation will bring us to a characteristic equation of the fourth degree. For stability, all coefficients of this equation must be positive and condition (l) must be satisfied.[1]

[1] A very complete bibliography on governors is given in "Encyklopädie der Mathematischen Wissenschaften," vol. 4_2. See article by R. von Mises, Dynamische Probleme der Maschinenlehre. See also Tolle, Die Regelung der Kraftmaschinen.

Example: Investigate the stability of motion of a pendulum of length l and mass m (Fig. 240) rotating with constant angular velocity ω about its vertical axis. Neglect the mass of the rod.

Solution: Assume that during the steady state of rotation about the vertical axis, a small angle of inclination θ is given to the pendulum as shown and let $\dot{\phi}$ denote the corresponding variation in the uniform angular velocity ω. Then the square of the total velocity of point C is

$$v^2 = l^2\theta^2(\omega + \dot{\phi})^2 + l^2\dot{\theta}^2;$$

and within the limits of small quantities of second order, the expressions for kinetic and potential energy are

$$T = \tfrac{1}{2}ml^2(\omega^2\theta^2 + \dot{\theta}^2),$$
$$V = \tfrac{1}{2}mgl\theta^2.$$

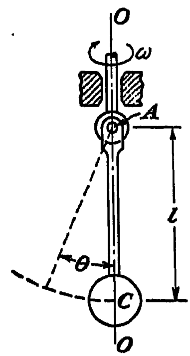

FIG. 240.

Substituting in the Lagrangian equation (155), we obtain the equation of motion

$$\ddot{\theta} + \left(\frac{g}{l} - \omega^2\right)\theta = 0. \tag{s}$$

This equation represents simple harmonic oscillations of the pendulum if

$$\frac{g}{l} - \omega^2 > 0. \tag{t}$$

Thus, when condition (*t*) is satisfied, the steady state of rotation is stable. It becomes unstable when

$$\frac{g}{l} - \omega^2 < 0.$$

If we assume that there is viscous damping proportional to $\dot{\theta}$, the equation of motion becomes

$$\ddot{\theta} + 2n\dot{\theta} + \left(\frac{g}{l} - \omega^2\right)\theta = 0.$$

Using the notation $p^2 = \omega^2 - (g/l)$ and assuming $\theta = Ce^{rt}$, we obtain the characteristic equation

$$r^2 + 2nr - p^2 = 0,$$

the roots of which are

$$r = -n \pm \sqrt{n^2 + p^2}.$$

We see that if p^2 is positive, one of the roots is positive, indicating that the steady motion is unstable if

$$\frac{g}{l} - \omega^2 < 0.$$

The introduction of friction does not change the condition of instability.

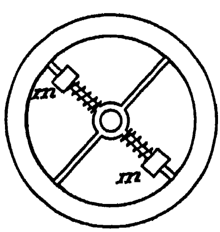

FIG. 241.

Example: Investigate the stability of steady rotation of a flywheel carrying two equal masses attached to the hub by springs and free to slide along two spokes as shown in Fig. 241. Assume that the wheel is in a horizontal plane.

Solution: Let I be the moment of inertia of the flywheel, ω its angular velocity in steady motion, s the corresponding radial distance of each mass m, a the radial distance of each mass when the springs are unstressed, and k the spring constants. Defining

small changes in the radial distances s by x and the corresponding variation in the angular velocity ω by ϕ, the expressions for kinetic and potential energy are

$$T = \tfrac{1}{2}I(\omega + \dot\phi)^2 + m[(s + x)^2(\omega + \dot\phi)^2 + \dot x^2],$$
$$V = k(s - a + x)^2 - k(s - a)^2.$$

The equations of motion, after omitting terms of higher order than the first in the small quantities ϕ and x, are

$$\left.\begin{aligned}(I + 2ms^2)\ddot\phi + 4ms\omega\dot x &= 0,\\ m\ddot x - m\omega^2(s + x) - 2m\omega s\dot\phi + k(s - a + x) &= 0.\end{aligned}\right\} \qquad (u)$$

Substituting in these equations $x = \dot x = \dot\phi = 0$, we obtain

$$m\omega^2 s = k(s - a). \qquad (v)$$

This equation states that the centrifugal forces of the masses m balance the spring forces in steady motion. Substituting $\dot\phi = C_1e^{rt}$, $x = C_2e^{rt}$ into Eqs. (u), and using Eq. (v), we obtain

$$(I + 2ms^2)rC_1 + 4ms\omega rC_2 = 0,$$
$$-2m\omega sC_1 + (mr^2 - m\omega^2 + k)C_2 = 0.$$

The characteristic equation is

$$(I + 2ms^2)r(mr^2 - m\omega^2 + k) + 8m^2s^2\omega^2r = 0.$$

This equation will have a positive root and the steady motion becomes unstable if

$$-m\omega^2 + k + \frac{8m^2s^2\omega^2}{I + 2ms^2} < 0$$

or

$$m\omega^2\left(1 - \frac{8ms^2}{I + 2ms^2}\right) > k. \qquad (w)$$

42. Variable-speed Dampers.—We have already seen in Art. 36 how forced vibrations of a system produced by a pulsating force can be eliminated by a pendulum tuned to the frequency of the disturbing force. However, such a device was seen to work satisfactorily only if the impressed frequency remains constant, and its practical application is thereby limited to synchronous machines. In the case of an internal-combustion engine, we may have forced torsional vibrations produced by a pulsating torque the frequency of which will be proportional to the variable speed of rotation of the engine. In such case, a pendulum like that shown in Fig. 238 of the preceding article can be made to act as a dynamical damper that will work for all speeds of rotation of the engine.

To illustrate, let us assume that a pulsating torque $A \sin n\omega t$ having a frequency proportional to ω acts on the rotating disk in Fig. 238. Then, instead of Eqs. (c), on page 314, the equations of motion for the system become

$$\left.\begin{aligned}I_0\ddot\theta + ml(l + r)\ddot\phi &= A \sin n\omega t,\\ \frac{l + r}{l}\ddot\theta + \ddot\phi + \frac{r\omega^2}{l}\phi &= 0.\end{aligned}\right\} \qquad (a)$$

A particular solution of these equations, representing forced vibrations, has the form

$$\theta = C \sin n\omega t, \qquad \phi = B \sin n\omega t.$$

Substituting into Eqs. (a), we obtain the amplitudes B and C, and the forced vibrations are

$$\left.\begin{aligned} \theta &= -\frac{A \sin n\omega t}{n^2\omega^2 I_0\left[1 - \dfrac{m(l+r)^2}{I_0\left(1 - \dfrac{r}{n^2 l}\right)}\right]}, \\ \phi &= \frac{A(l+r)\sin n\omega t}{n^2\omega^2 I_0 l\left(1 - \dfrac{r}{n^2 l}\right)\left[1 - \dfrac{m(l+r)^2}{I_0\left(1 - \dfrac{r}{n^2 l}\right)}\right]}. \end{aligned}\right\} \tag{b}$$

For comparison we consider also the disk alone, without any pendulum, under the action of the same pulsating torque. In such case, the equation of motion is

$$I\ddot{\theta} = A \sin n\omega t,$$

and we obtain

$$\theta = -\frac{A}{In^2\omega^2} \sin n\omega t. \tag{c}$$

Comparing this with the first of Eqs. (b) above, we conclude that the action of the pendulum is equivalent to increasing the moment of inertia of the disk by the amount

$$m(l+r)^2 + \frac{m(l+r)^2}{\left(\dfrac{r}{n^2 l} - 1\right)} = \frac{m(l+r)^2}{1 - \dfrac{n^2 l}{r}}. \tag{d}$$

It is seen that by taking

$$\frac{n^2 l}{r} = 1, \tag{e}$$

we obtain a perfectly tuned pendulum, the action of which is equivalent to that of a flywheel of infinite moment of inertia. In this case, the steady motion of the disk will not be affected by the pulsating torque while the pendulum will perform the forced vibrations

$$\phi = -\frac{A \sin n\omega t}{mn^2\omega^2 l(l+r)}. \tag{f}$$

We see that this vibration has the phase angle π with respect to the pulsating torque and counteracts it.

In the case of a multicylinder engine, the angular frequency $n\omega$ of the pulsating torque is usually several times larger than ω and the length

of the pendulum, as obtained from Eq. (e), becomes small. Practically, such a pendulum can be built in the form of a cylinder rolling inside a circular hole[1] as shown in Fig. 242. From Eq. (f) we see that if the mass of the roller is too small, its amplitude of vibration becomes too large and the theory developed on the assumption of small vibrations is not sufficiently accurate. The influence of friction and the possibility of slipping of the roller must also be considered in practical applications.[2]

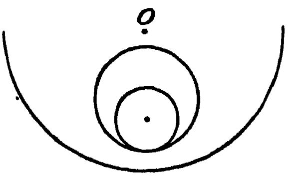

FIG. 242.

By proper design, the spring-suspended masses m on the flywheel in Fig. 241, page 319, can also be made to serve as a dynamical damper. Assume, for example, that a pulsating torque $M \sin n\omega t$ acts on the flywheel. Then, instead of Eqs. (u) on page 320, we have

$$\left.\begin{aligned} (I + 2ms^2)\ddot{\phi} + 4ms\omega\dot{x} &= M \sin n\omega t, \\ m\ddot{x} - m\omega^2(s + x) - 2m\omega s\phi + k(s - a + x) &= 0. \end{aligned}\right\} \tag{g}$$

Let us investigate now under what condition the pulsating torque will not produce fluctuations in the angular velocity of the flywheel. Substituting $\phi = 0$, we obtain

$$\left.\begin{aligned} 4ms\omega\dot{x} &= M \sin n\omega t, \\ m\ddot{x} + (k - m\omega^2)x &= 0. \end{aligned}\right\} \tag{h}$$

If $(k - m\omega^2)$ is positive, the second equation represents simple harmonic vibrations of the angular frequency

$$p = \sqrt{\frac{k}{m} - \omega^2}.$$

Selecting the spring constant so as to make

$$\sqrt{\frac{k}{m} - \omega^2} = n\omega \qquad \text{or} \qquad \frac{k}{m} = (n^2 + 1)\omega^2 \tag{i}$$

and taking

$$x = -\frac{M}{4msn\omega^2}\cos n\omega t,$$

we satisfy both of Eqs. (h), which indicates that the sliding masses counteract the pulsating torque and work as a damper if the spring constant satisfies condition (i).

[1] See papers by B. Salomon in *Proc. Fourth Intern. Cong. Applied Mech.*, Cambridge, England, 1934, and by E. S. Taylor, *Trans. Soc. Automotive Eng.*, p. 81, 1936.

[2] These questions are discussed by J. P. Den Hartog, "S. Timoshenko 60th Anniversary Volume," p. 17, Macmillan, 1938. See also paper by A. H. Shioh, *J. Aeronaut. Sci.*, vol. 9, p. 337, 1942.

To make the device work efficiently at various speeds, we have to use springs the stiffness of which varies with the extension.[1] In such case, the spring force will no longer be proportional to the elongation $s - a$ but will be some other function $f(s)$ and Eq. (v) on page 320 will be replaced by the following equation:

$$m\omega^2 s = f(s). \tag{j}$$

Instead of a spring constant k, we now have a variable stiffness defined by the magnitude of the derivative $df(s)/ds$, and condition (i) above will be replaced by

$$\frac{df(s)}{ds} = m(n^2 + 1)\omega^2. \tag{k}$$

Eliminating ω^2 from Eqs. (j) and (k), we obtain

$$\frac{df(s)}{ds} = (n^2 + 1)\frac{f(s)}{s},$$

and by integration

$$f(s) = Cs^{n^2+1}. \tag{l}$$

Assuming that for a certain prescribed angular velocity ω_0, the radial distance of each mass m is equal to b, we find

$$f(b) = m\omega_0^2 b = Cb^{n^2+1},$$
$$C = m\omega_0^2 b^{-n^2}.$$

Substituting into Eq. (l), we obtain the following equation defining the force in the spring:

$$f(s) = m\omega_0^2 b\left(\frac{s}{b}\right)^{n^2+1}. \tag{m}$$

In this theoretical discussion, the frictional resistance to sliding of the masses along the spokes of the wheel was neglected. Practically, such resistance is always present and it is difficult to eliminate it, since due to Coriolis forces, there must exist a considerable lateral pressure between the masses m and the spokes during oscillations.

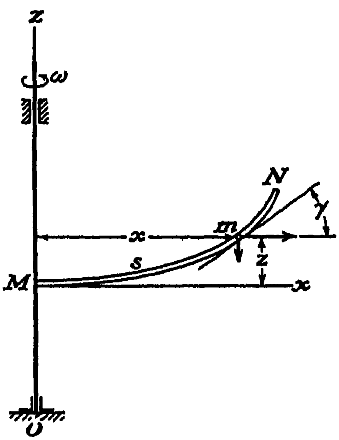

FIG. 243.

As a last example, let us consider the system shown in Fig. 243. A plane curvilinear tube MN attached to a vertical shaft rotates with constant angular velocity ω as shown. Inside the tube is a particle of mass m which can slide freely along its axis. Under proper conditions,

[1] The feasibility of using such springs and their application in damper devices is discussed by R. Grammel, *Ing. Arch.*, vol. I, p. 3, 1930.

this system can be made to function as a damper, as we shall now show. Let I be the moment of inertia, about the z-axis, of the shaft and tube together, s the displacement of the particle m measured along the axis of the tube, and x, z its rectangular coordinates as shown. For a given shape of the tube and a given angular velocity ω, the position of the particle m in steady motion can be readily obtained from the equilibrium condition

$$m\omega^2 x \cos\gamma = mg \sin\gamma. \tag{n}$$

Observing that

$$\cos\gamma = \frac{dx}{ds} = x' \qquad \text{and} \qquad \sin\gamma = \frac{dz}{ds} = z'$$

and using the subscript zero for quantities referred to the equilibrium position of the particle, Eq. (n) becomes

$$\omega^2 x_0 x_0' = g z_0' = g\sqrt{1 - x_0'^2}. \tag{o}$$

Assume now a small displacement η from the equilibrium position of the particle and a small fluctuation ϕ in the angular velocity of the shaft. Then when $s = s_0 + \eta$, the angular velocity is $\omega + \phi$ and the expressions for kinetic and potential energy of the system become

$$T = \tfrac{1}{2}(I + mx^2)(\omega + \phi)^2 + \tfrac{1}{2}m\dot{s}^2,$$
$$V = mgz = mg(z_0 + z_0'\eta).$$

Substituting these expressions in Lagrangian equations (155), page 314, and observing that

$$x = x_0 + \eta x_0', \qquad x' = x_0' + x_0''\eta, \qquad \dot{x} = \frac{dx}{dt} = \frac{dx}{ds}\frac{ds}{dt} = x_0'\dot{\eta},$$
$$z = z_0 + z_0'\eta, \qquad z' = z_0' + z_0''\eta,$$

we obtain, after omitting powers and products of small quantities,

$$\left.\begin{aligned} (I + mx_0^2)\ddot{\phi} + 2m\omega x_0 x_0'\dot{\eta} &= 0,\\ \ddot{\eta} + \eta[gz_0'' - \omega^2(x_0'^2 + x_0x_0'')] - 2\omega x_0 x_0'\dot{\phi} &= 0. \end{aligned}\right\} \tag{p}$$

Since x_0, x_0', x_0'', z_0'' are defined by the known shape of the axis of the tube, Eqs. (p) represent two linear equations with constant coefficients and the stability of motion can be investigated in the usual way (see Art. 41).

To investigate the work of the device as a damper, we assume that a pulsating torque $M \sin n\omega t$ is acting on the shaft. This pulsating torque will appear on the right-hand side of the first of Eqs. (p). Then putting $\dot{\phi} = 0$, we obtain

$$\left.\begin{aligned} 2m\omega x_0 x_0'\dot{\eta} &= M \sin n\omega t,\\ \ddot{\eta} + \eta[gz_0'' - \omega^2(x_0'^2 + x_0x_0'')] &= 0. \end{aligned}\right\} \tag{q}$$

These equations are similar to Eqs. (h) of the preceding example and we conclude that for tuning the damper, we must fulfill the condition

$$gz_0'' - \omega^2(x_0'^2 + x_0x_0'') = n^2\omega^2.$$

Observing that

$$z' = \sqrt{1 - x'^2}, \qquad z'' = -\frac{x'x''}{\sqrt{1 - x'^2}},$$

the condition for tuning becomes

$$\frac{gx_0'x_0''}{\sqrt{1 - x_0'^2}} + \omega^2(x_0'^2 + x_0x_0'') + n^2\omega^2 = 0. \tag{r}$$

Eliminating ω^2 from Eqs. (r) and (o), we obtain the equation of the center line of the tube for which the tuning condition is satisfied for all values of ω:

$$xx'' + (1 - x'^2)(x'^2 + n^2) = 0. \tag{s}$$

To minimize friction, liquid oscillating in a rotating tube can be utilized for damping undesirable vibrations.[1]

[1] This type of damper was suggested by E. Meissner, see *Proc. Third Intern. Congr. Applied Mech.*, vol. 3, p. 199, Stockholm, 1930. It represents the first variable speed damper.

CHAPTER V

ROTATION OF A RIGID BODY ABOUT A FIXED POINT

43. Kinematics of Rotation about a Fixed Point.—The rotation of a rigid body about a fixed axis represents a very simple problem in dynamics; but when the axis of rotation is also in motion, the problem becomes much more complicated. The special case where the axis of rotation always passes through a fixed point is of particular practical interest in connection with the theory of the *gyroscope*[1] which has many important

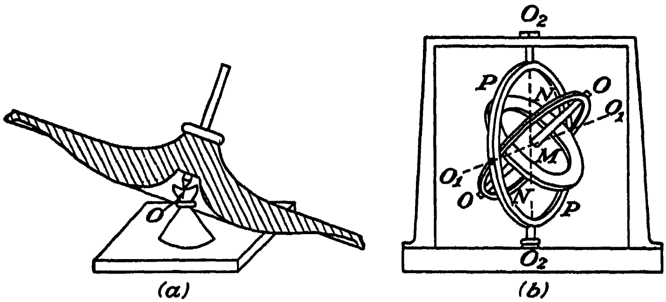

FIG. 244.

technical applications. To make a body rotate about a fixed point, such arrangements as those shown in Fig. 244 can be used. In Fig. 244*a*, a body of revolution is supported so that the pivot point O remains practically immovable. This, of course, is the type of support encountered in the case of a spinning top. In Fig. 244*b*, a circular disk M can rotate freely about its geometric axis OO supported by bearings in the frame NN. This frame, in turn, can rotate about the axis O_1O_1 pivoted in the frame PP which is free to rotate about the fixed axis O_2O_2. All

[1] In 1852, Leon Foucault presented a device to prove rotation of the earth and named it the gyroscope, derived from the Greek and meaning indicator of rotation. In this chapter, we shall give only brief discussions of some of the technical applications of gyroscopes. For further details, special books on gyroscopes can be consulted. See (1) F. Klein und A. Sommerfeld, "Theorie des kreisels," 1897–1910; (2) Andrew Gray, "A Treatise on Gyrostatics and Rotational Motion," 1918; (3) R. Grammel, "Der Kreisel," 1920; (4) A. N. Krylov and U. A. Krutkov, "Theory of Gyroscopes and Some of Their Technical Applications," 1932 (Russian). (5) Oscar Martienssen, Kreiselbewegung and Technische Anwendung des Kreisels, in "Handbuch der physikalischen und technischen Mechanik," vol. 2, pp. 405, 444, 1930.

three axes intersect in one point, and the corresponding point of the disk M remains fixed in space.[1] This type of support is the one most commonly encountered in gyroscopic applications.

To define the position of a rigid body rotating about a fixed point, we need three coordinates. In general, we shall find it most convenient to take the three angles ψ, θ, and ϕ, as shown in Fig. 245. The origin of

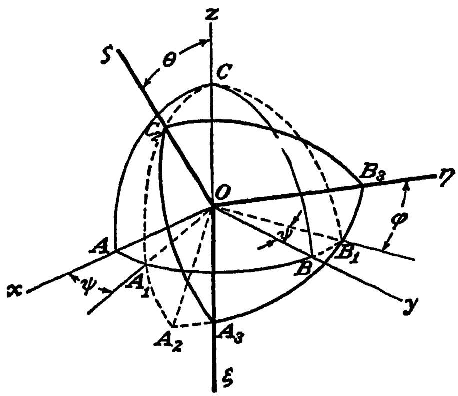

FIG. 245.

coordinates O is the fixed point about which the body rotates, and the orthogonal axes x, y, z through this point are fixed in space, while the axes ξ, η, ζ through the same point rotate with the body. To show that the three angles mentioned above completely define the position of the rotating body, assume that the moving axes ξ, η, ζ initially coincide with the fixed axes x, y, z. Then rotating the body about the z-axis through the angle ψ, we bring it to the position A_1B_1C. Next, we rotate the body about the so-called *nodal axis* OB_1, through the angle θ and bring it into the position $A_2B_1C_2$. Finally, we make rotation of the body with respect to the ζ-axis through the angle ϕ and bring the moving system of coordinate axes into the final position $\xi\eta\zeta$ as shown. This shows that the three angles ψ, θ, ϕ completely define the position of a body rotating about a fixed point O and can be taken as coordinates. In defining the signs of rotation, we use the right-hand screw rule so that the angles ψ, θ, and ϕ, as shown in Fig. 245, represent positive rotations with respect to Oz, OB_1, and $O\zeta$, respectively.

Sometimes, instead of the angles ψ, θ, ϕ shown in Fig. 245, a slightly different system of coordinates may be taken. Instead of the angle θ, we may take its complement $\beta = (\pi/2) - \theta$, representing the angle of

[1] Strictly speaking, we have in this case a system of several moving bodies; but in many practical cases, the masses of the rotating frames can be neglected and we can consider the motion of the disk M alone.

inclination of the moving ζ-axis to the fixed xy-plane. Likewise, instead of ϕ, we may take $\alpha = \phi - (\pi/2)$, which is the angle that the moving ξ-axis makes with the nodal axis OB_1. Regarding the nodal axis, it should be noted that it is defined by the line of intersection of the moving $\xi\eta$-plane with the fixed xy-plane. Thus during motion of a body about a fixed point O, the nodal axis remains always in the fixed xy-plane and rotates about the z-axis with angular velocity $\dot{\psi}$.

After selecting the coordinates, let us consider displacements, velocities, and accelerations of any point of a body rotating about a fixed point O. Any displacement of such a body can be visualized as a rotation about a properly selected axis. To show this, consider a sphere having its center at the fixed point O and moving with the body (Fig. 246). The displacement of this sphere is completely defined if we know the displacements of any two of its points, say A and B, provided the straight line AB does not pass through O. Assume that the displacement is such that point A moves to A_1 while point B moves to B_1.

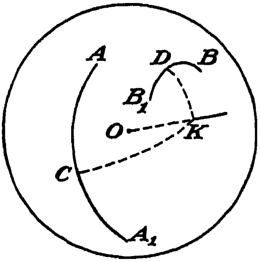

FIG. 246.

Taking two planes COK and DOK passing through O and bisecting the circular arcs AA_1 and BB_1, we obtain, as their line of intersection, the axis OK. Rotation with respect to this axis will accomplish the assumed displacements. This follows from the fact that the spherical triangles ABK and A_1B_1K are equal. Hence if, by rotation about OK, the side BK is brought into the position B_1K, the side AK will coincide with A_1K.

Any continuous motion of a body about a fixed point O can be accomplished by dividing the time into infinitesimal elements and considering motion of the body during each element of time as rotation about the corresponding *instantaneous axis of rotation* such as OK in Fig. 246. During motion, the position of the instantaneous axis both in the body and in space is gradually changing. To show this, we take the case of a cone rolling on a plane and rotating about its fixed vertex O (Fig. 247). At any instant, the motion of the cone consists of rotation about the line of tangency OA, which is the instantaneous axis in this case. As the cone rolls, various lines of its surface come into contact with the plane, and we see that consecutive positions of the instantaneous axis in the moving cone form its lateral surface. At the same time, considering consecutive positions of the instantaneous axis in space, we see that they form the plane on which the cone is rolling.

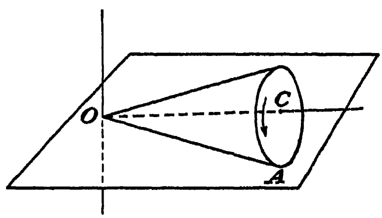

FIG. 247.

The above example can be generalized. Instead of a circular cone rolling on a plane, we can take a cone of arbitrary shape and can make it roll on the surface of another cone as shown in Fig. 248. We can assume also that the rolling cone AOB is not a physical body but a geometrical surface formed by consecutive positions of the instantaneous axis in the moving body of any shape as indicated by dotted lines. Likewise, the stationary cone is formed by successive positions of the instantaneous axis in space, and the motion of the body is visualized by rolling the cone connected with the body on the cone fixed in space. These cones are sometimes called *body cone* and *space cone*, respectively. By varying the shapes of both cones, we can obtain all possible motions of a rigid body about a fixed point.[1]

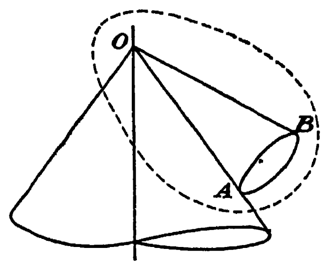

FIG. 248.

Referring now to Fig. 249a, assume that OA is the direction of the instantaneous axis at any instant t and that during the time from t to $t + \Delta t$, the body rotates by an angle $\Delta\alpha$. Then the angular velocity ω of the body at the instant t is

$$\omega = \lim_{\Delta t \to 0} \left[\frac{\Delta\alpha}{\Delta t}\right].$$

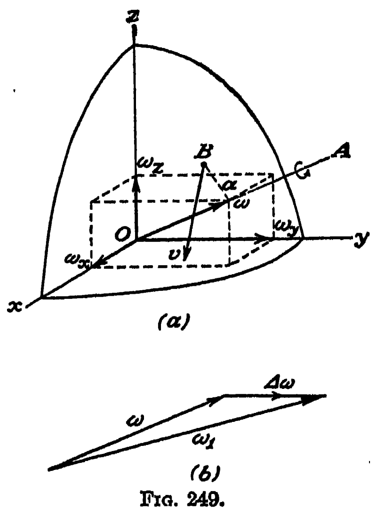

FIG. 249.

We will represent this angular velocity by a vector directed along the instantaneous axis in accordance with the right-hand screw rule.

During motion, both the direction and magnitude of the angular velocity of a body may change. In Fig. 249b, let ω be the angular velocity of the body at any instant t and ω_1 its angular velocity at the instant $t + \Delta t$. Then the geometrical difference $\Delta\omega$ of the two vectors represents the change in angular velocity during the interval of time Δt, and we define the angular acceleration of the body by the quantity

$$\epsilon = \lim_{\Delta t \to 0} \left[\frac{\Delta\omega}{\Delta t}\right].$$

We ascribe to this quantity the direction coinciding with the direction of

[1] This manner of visualizing the motion of a body about a fixed point is due to L. Poinsot, *J. math.*, vol. 16, 1851.

$\Delta\omega$ in Fig. 249*b*, so that the angular acceleration of a body is represented by the velocity of the end of the vector representing its angular velocity ω. The magnitude and direction of the angular acceleration depend on the rate of change in magnitude and direction of the angular velocity. If the direction of the angular velocity is constant and the body rotates about a fixed axis, the vector representing angular acceleration has the same direction as the angular velocity when this velocity is increasing and the opposite direction when the angular velocity is decreasing. If the angular velocity has constant magnitude but its direction is changing, as in the case of the cone, rolling with constant speed around point O in Fig. 247, the angular acceleration will be represented by a vector perpendicular to the vector ω.

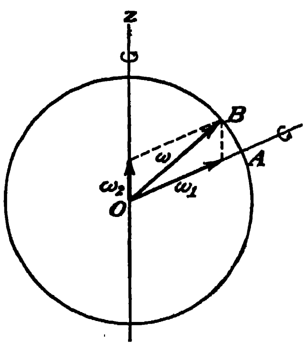

FIG. 250.

Referring again to Fig. 249*a*, where OA is the instantaneous axis of rotation of a body at any instant t, we see that the velocity of any point B at the distance a from the instantaneous axis is $v = \omega a$. This velocity will be perpendicular to the plane AOB and directed so that between it and the angular velocity ω there will be the same relation as between rotation and translation of a right-hand screw. We see that the velocity vector v can be considered as representing the moment of the angular velocity vector ω with respect to point B. This relation will be useful in our further discussion when it will be necessary to calculate projections on the coordinate axes of the rotational velocity v.

Sometimes we have to consider two simultaneous rotations of a body. This happens, for example, if we wish to discuss the influence of rotation of the earth on the motion of a body. Referring to Fig. 250, assume that a body placed at A on the surface of the earth rotates about the axis OA with angular velocity ω_1. At the same time, it rotates with the earth about the z-axis with angular velocity ω_2. The rotational velocities of any point of the body due to these two angular velocities will be obtained as the moments of the vectors ω_1 and ω_2 about that point. But we know that the sum of these two moments is equal to the moment of the geometrical sum ω of the two vectors ω_1 and ω_2, and we conclude accordingly that the resultant motion of the body is a rotation about the axis OB with angular velocity ω equal to the geometrical sum of the vectors ω_1 and ω_2. Thus, if we have several simultaneous angular velocities, we can sum them up geometrically as we sum up forces. We can also resolve the angular velocity ω of a body into components and consider,

separately, rotation corresponding to each component. Sometimes we take, for these components, the projections ω_x, ω_y, ω_z of the resultant angular velocity on the fixed coordinate axes x, y, z and sometimes the projections ω_ξ, ω_η, ω_ζ on the moving axes ξ, η, ζ.

Using such resolution of the angular velocity ω into its rectangular components, we can readily write expressions for the x-, y-, z-projections of the rotational velocity v of any point B of a body rotating about a fixed point O (Fig. 249). Since the rotational velocity v can be considered as the moment of the vector ω about point B, its projection on any axis will be equal to the moment of ω about a parallel axis passing through B. Thus, treating the components of ω as components of a force, we obtain the following expressions for the projections of the rotational velocity v on the immovable coordinate axes:

$$\left.\begin{aligned} v_x &= \omega_y z - \omega_z y, \\ v_y &= \omega_z x - \omega_x z, \\ v_z &= \omega_x y - \omega_y x. \end{aligned}\right\} \tag{156}$$

In a similar manner, the projections of the velocity v on moving axes ξ, η, ζ, (not shown in Fig. 249) will be

$$\left.\begin{aligned} v_\xi &= \omega_\eta \zeta - \omega_\zeta \eta, \\ v_\eta &= \omega_\zeta \xi - \omega_\xi \zeta, \\ v_\zeta &= \omega_\xi \eta - \omega_\eta \xi. \end{aligned}\right\} \tag{157}$$

In these equations, ξ, η, ζ are the coordinates of the point B with respect to the moving axes and ω_ξ, ω_η, ω_ζ are the projections of the angular velocity ω on these axes.

PROBLEMS

151. Find the resultant angular velocity of the driving axle of a locomotive rounding a curve at 50 m.p.h. if the diameter of the driving wheels is 6 ft. and the radius of the curve is 1,000 ft.

152. Taking into consideration rotation of the earth, find the absolute angular velocity of a rotor making 1,800 r.p.m. if the latitude of the place is 45 deg. and the axis of the rotor is in the direction (*a*) of a tangent to the meridian, (*b*) of a perpendicular to the meridian.

153. The cone in Fig. 247 has an apex angle of 20 deg. and rolls uniformly on the horizontal plane, making 10 r.p.m. about point O. Find the magnitude and direction of the resultant angular velocity ω of the cone and also the angular velocity about its own geometric axis.

44. Equations of Motion about a Fixed Point.—To derive the equations of motion for a body rotating about a fixed point, we shall use the principle of angular momentum (see page 121). Having Eqs. (156) and (157) representing projections of the velocity v of any point of the body

on the immovable axes x, y, z and also on the moving axes ξ, η, ζ, there will be no difficulty in calculating the angular momentum of the body with respect to any of these axes. For our further discussion, we shall now calculate the angular momenta with respect to the moving axes ξ, η, ζ. Taking an element of the body of mass dm and having coordinates ξ, η, ζ, we find that its moment of momentum with respect to the ξ-axis is

$$dm(v_\zeta \eta - v_\eta \zeta).$$

Substituting for v_ζ and v_η their expressions from Eqs. (157) and extending the summation over the entire volume of the body, we obtain the following expression for the angular momentum of the body about the ξ-axis:

$$\begin{aligned} H_\xi &= \int(\omega_\xi \eta - \omega_\eta \xi)\eta\, dm - \int(\omega_\zeta \xi - \omega_\xi \zeta)\zeta\, dm \\ &= \omega_\xi \int(\eta^2 + \zeta^2)dm - \omega_\eta \int \xi\eta\, dm - \omega_\zeta \int \xi\zeta\, dm. \end{aligned}$$

Similar expressions can be written for the angular momenta with respect to the η- and ζ-axes. The integrals entering in these expressions represent the moments of inertia and the products of inertia of the body with respect to the axes ξ, η, ζ, moving with the body. Using for these quantities the notations

$$\begin{aligned} &I_\xi = \int(\eta^2 + \zeta^2)dm, \qquad I_\eta = \int(\xi^2 + \zeta^2)dm, \qquad I_\zeta = \int(\xi^2 + \eta^2)dm, \\ &I_{\xi\eta} = \int \xi\eta\, dm, \qquad I_{\eta\zeta} = \int \eta\zeta\, dm, \qquad I_{\xi\zeta} = \int \xi\zeta\, dm, \end{aligned}$$

we express the angular momenta of the body with respect to the moving axes in the following form:

$$\left.\begin{aligned} H_\xi &= I_\xi \omega_\xi - I_{\xi\eta}\omega_\eta - I_{\xi\zeta}\omega_\zeta, \\ H_\eta &= I_\eta \omega_\eta - I_{\eta\zeta}\omega_\zeta - I_{\xi\eta}\omega_\xi, \\ H_\zeta &= I_\zeta \omega_\zeta - I_{\xi\zeta}\omega_\xi - I_{\eta\zeta}\omega_\eta. \end{aligned}\right\} \tag{158}$$

In our further discussion, we usually shall select for ξ, η, ζ, the principal axes of the body at point O. The products of inertia then vanish, and we obtain

$$H_\xi = I_\xi \omega_\xi, \qquad H_\eta = I_\eta \omega_\eta, \qquad H_\zeta = I_\zeta \omega_\zeta. \tag{159}$$

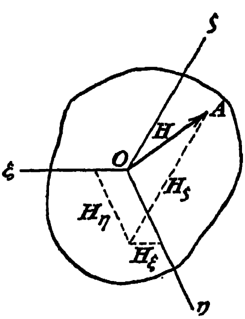

FIG. 251.

Having the angular momenta of the body with respect to the moving axes, we obtain the resultant angular momentum H with respect to the fixed point O as the geometric sum of H_ξ, H_η, H_ζ. In Fig. 251, this resultant angular momentum is represented by the vector $\overline{OA}$. The principle of angular momentum now states that the rate of change of angular momentum of the body rotating about a fixed point O is equal to the moment of all forces acting on the body with respect to the same point. This rate of change of angular momen-

tum is represented by the velocity of the end A of the vector $\overline{OA}$. This velocity can be considered as the geometric sum of two velocities: (1) the velocity of point A with respect to the coordinate axes ξ, η, ζ and (2) the velocity of the point A moving together with the $\xi\eta\zeta$ coordinate system. Since the coordinates of A with respect to the moving axes are H_ξ, H_η, H_ζ, respectively, the components of the velocity of point A with respect to these axes are

$$\frac{dH_\xi}{dt}, \qquad \frac{dH_\eta}{dt}, \qquad \frac{dH_\zeta}{dt}. \tag{a}$$

To get the components of the velocity of point A due to its motion together with the $\xi\eta\zeta$ coordinate system, we use Eqs. (157). Substituting into these equations the coordinates H_ξ, H_η, H_ζ of the point A instead of ξ, η, ζ, we obtain

$$\omega_\eta H_\zeta - \omega_\zeta H_\eta, \qquad \omega_\zeta H_\xi - \omega_\xi H_\zeta, \qquad \omega_\xi H_\eta - \omega_\eta H_\xi. \tag{b}$$

Adding together the corresponding components (*a*) and (*b*), we obtain the projections on the ξ-, η-, ζ-axes of the total velocity of point A. From the principle of angular momentum, these projections must be equal to the moments, with respect to the same axes, of all external forces acting on the body. In this way, we obtain the following equations of motion of a body rotating about a fixed point:

$$\left.\begin{aligned}
\frac{dH_\xi}{dt} + \omega_\eta H_\zeta - \omega_\zeta H_\eta &= M_\xi,\\
\frac{dH_\eta}{dt} + \omega_\zeta H_\xi - \omega_\xi H_\zeta &= M_\eta,\\
\frac{dH_\zeta}{dt} + \omega_\xi H_\eta - \omega_\eta H_\xi &= M_\zeta.
\end{aligned}\right\} \tag{160}$$

Assuming that ξ, η, ζ are principal axes of the body and using Eqs. (159), we represent the above equations of motion in the following form:

$$\left.\begin{aligned}
I_\xi \frac{d\omega_\xi}{dt} + (I_\zeta - I_\eta)\omega_\eta\omega_\zeta &= M_\xi,\\
I_\eta \frac{d\omega_\eta}{dt} + (I_\xi - I_\zeta)\omega_\xi\omega_\zeta &= M_\eta,\\
I_\zeta \frac{d\omega_\zeta}{dt} + (I_\eta - I_\xi)\omega_\xi\omega_\eta &= M_\zeta.
\end{aligned}\right\} \tag{161}$$

Here I_ξ, I_η, I_ζ are the moments of inertia of the body with respect to principal axes through the fixed point O, ω_ξ, ω_η, ω_ζ are the projections of its angular velocity on the ξ-, η-, ζ-axes; and M_ξ, M_η, M_ζ are the moments of external forces with respect to the same axes. Equations (161) were first derived by Euler and are called *Euler's equations.*

In the derivation of Eqs. (160), we assumed that the ξ-, η-, ζ-axes are rigidly connected with the moving body, but it is sometimes advantageous to select other coordinates ξ', η', ζ', such that the trihedron $O\xi'\eta'\zeta'$ rotates about O with some angular velocity ω' different from the angular velocity ω of the body. Denoting then by H_ξ', H_η', H_ζ' the angular momenta of the body with respect to these new axes and by ω_ξ', ω_η', ω_ζ', the corresponding components of the angular velocity ω', we obtain, instead of Eqs. (160), the following equations:

$$\left.\begin{aligned}\frac{dH_\xi'}{dt} + \omega_\eta' H_\zeta' - \omega_\zeta' H_\eta' &= M_\xi',\\ \frac{dH_\eta'}{dt} + \omega_\zeta' H_\xi' - \omega_\xi' H_\zeta' &= M_\eta',\\ \frac{dH_\zeta'}{dt} + \omega_\xi' H_\eta' - \omega_\eta' H_\xi' &= M_\zeta'.\end{aligned}\right\} \tag{162}$$

In this case, the selected axes $\xi'\eta'\zeta'$ are moving with respect to the body, and the moments of inertia and the products of inertia may not be constant, as in Eqs. (161), but may vary with time. Equations (162) will be called *modified Euler's equations.*

Since the position of a body, rotating about a fixed point O is defined by the three angles ψ, θ, ϕ, as shown in Fig. 245, the variable quantities in Eqs. (161) and (162) must also be represented in terms of these three angles and their derivatives with respect to time. Thus we obtain a system of three simultaneous differential equations, which represent the required equations of motion for a body rotating about a fixed point O. In subsequent articles, we shall consider the solution of these equations for various cases of gyroscopic motion, but first we may find it of interest to mention two very simple particular cases.

As a first case, we see at once that Eqs. (161) will be satisfied if we take $\omega_\xi = \omega_\eta = 0$, $M_\xi = M_\eta = 0$ and select ω_ζ so as to satisfy the equation

$$I_\zeta \frac{d\omega_\zeta}{dt} = M_\zeta. \tag{c}$$

From this equation, we conclude that if a body free to rotate about a fixed point O is acted upon by a couple the plane of which is normal to a principal axis through that point and any initial angular velocity is about this axis, then the motion is the same as rotation about a fixed axis.[1]

In another particular case where $I_\xi = I_\eta = I_\zeta, = I$, Eqs. (161) reduce to

$$I\frac{d\omega_\xi}{dt} = M_\xi, \qquad I\frac{d\omega_\eta}{dt} = M_\eta, \qquad I\frac{d\omega_\zeta}{dt} = M_\zeta. \tag{d}$$

[1] See the authors' "Engineering Mechanics," 2d ed., p. 382.

In this case, the three equations of motion are independent and rotation about any principal axis depends only on the initial angular velocity and the moment of external forces about that axis. Thus, for example, if the earth be considered as a homogeneous sphere, we conclude that the effect of an external moment about an equatorial diameter would be independent of the initial rotation of the earth about its polar axis.

45. Free Motion of a Gyroscope.—Let us consider now the motion of a symmetrical gyroscope having the form of a body of revolution as shown in Fig. 244*a*. The axis of revolution is called the *axis of the gyroscope* or the *axis of spin*, and the corresponding moment of inertia I is called the *axial moment of inertia*. With the point of support O as origin, we take again the system of coordinate axes ξ, η, ζ, moving together with the gyroscope as shown in Fig. 252.

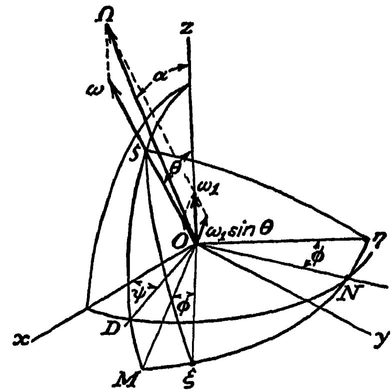

FIG. 252.

The ζ-axis coincides with the axis of the gyroscope so that $I_\zeta = I$. From symmetry,[1] we conclude that $I_\xi = I_\eta = I_1$, and in our further discussion this will be called the *equatorial moment of inertia*.

We shall now consider the motion of this symmetrical gyroscope, assuming that no external forces act upon it. The body shown in Fig. 244*a* represents such an example if the point of support O coincides with the center of gravity of the body. In this case, the gravity force is always balanced by the reaction at the point of support; and if some initial impulse be given to the body, it moves as if no forces were acting on it. In such case, Euler's equations (161) become

$$\left.\begin{aligned} I_1 \frac{d\omega_\xi}{dt} + (I - I_1)\omega_\eta\omega_\zeta &= 0, \\ I_1 \frac{d\omega_\eta}{dt} - (I - I_1)\omega_\xi\omega_\zeta &= 0, \\ I\frac{d\omega_\zeta}{dt} &= 0. \end{aligned}\right\} \qquad (163)$$

Referring to Fig. 252, we shall now show that the motion satisfying these equations can be described in the following way: The gyroscope,

[1] The characteristic properties of a symmetrical gyroscope depend only on the fact that $I_\xi = I_\eta$. If this condition is satisfied, we call the gyroscope symmetrical although its ζ-axis may not be an axis of geometrical symmetry. A three-blade propeller can be considered as a symmetrical gyroscope although it is not a body of revolution.

rigidly connected with the moving system of coordinates ξ, η, ζ, rotates with constant angular velocity ω about its axis $O\zeta$, making a constant angle θ with the immovable vertical z-axis. At the same time, the plane $zO\zeta$, containing the axis of the gyroscope, rotates with constant angular velocity ω_1 about the z-axis. Assuming that the moving $\xi\zeta$-plane initially coincides with the fixed xz-plane, we conclude that while the angle θ remains constant, the other two angles ψ and ϕ, defining the position of the gyroscope, will vary with time as represented by the equations

$$\psi = \omega_1 t, \qquad \phi = \omega t. \tag{a}$$

The gyroscope has two angular velocities, namely: ω, about its own ζ-axis and ω_1 about the z-axis. Projecting these velocities on the moving axes ξ, η, ζ, in Fig. 252, we obtain

$$\left.\begin{aligned} \omega_\xi &= -\omega_1 \sin\theta\cos\phi = -\omega_1\sin\theta\cos\omega t,\\ \omega_\eta &= \omega_1\sin\theta\sin\phi = \omega_1\sin\theta\sin\omega t,\\ \omega_\zeta &= \omega + \omega_1\cos\theta. \end{aligned}\right\} \tag{b}$$

Substituting these expressions into Eqs. (163), we see that the third equation is satisfied, since it was assumed that ω, ω_1, and θ are constants. Also, adding together the first two equations, we obtain

$$I\omega + (I - I_1)\omega_1\cos\theta = 0, \tag{164}$$

which requires that

$$\omega_1 = \frac{I\omega}{(I_1 - I)\cos\theta}. \tag{165}$$

We see that by assuming the angle θ constant and taking expressions (a) for ψ and ϕ, we can satisfy all three of the equations of motion (163), provided condition (164) is satisfied. This indicates that the assumed motion represents the required solution for the motion of a symmetrical gyroscope, free from the action of external forces. This motion of the gyroscope is called *regular precession*, and the angular velocity ω_1 is called the *velocity of precession*.

Let us consider now the angular momentum of the gyroscope during regular precession. The resultant angular velocity, which we shall denote by Ω, can be resolved into two components: $\omega + \omega_1\cos\theta$ along the ζ-axis and $-\omega_1\sin\theta$ along the axis OM as shown in Fig. 252. In the case of a symmetrical gyroscope, both OM and $O\zeta$ are principal axes, and hence the corresponding components of the angular momentum are

$$H_\zeta = I(\omega + \omega_1\cos\theta), \qquad H_m = -I_1\omega_1\sin\theta.$$

Observing, from Eq. (164), that

$$I(\omega + \omega_1\cos\theta) = I_1\omega_1\cos\theta,$$

we may write these components of angular momentum as follows:

$$\left.\begin{aligned} H_m &= -I_1\omega_1\sin\theta, \\ H_\zeta &= I_1\omega_1\cos\theta. \end{aligned}\right\} \tag{c}$$

From these expressions, we conclude that the resultant angular momentum H has the same direction as ω_1 and that its magnitude is

$$H = I_1\omega_1. \tag{166}$$

Thus, during regular precession of a symmetrical gyroscope, the resultant angular momentum vector H remains constant both in magnitude and direction as it should, since there are no external forces acting. It should be noted that during such motion, the four vectors representing, respectively, the resultant angular momentum H, the resultant angular velocity Ω, the velocity of precession ω_1, and the angular velocity ω of rotation of the gyroscope about its own axis, all remain in the rotating $zO\zeta$-plane as shown in Fig. 252.

Regarding the angle α between the vectors H and Ω, an important conclusion can be made by using Eq. (114), page 159, for the kinetic energy of a moving body, from which we obtain

$$2T = I_\xi\omega_\xi^2 + I_\eta\omega_\eta^2 + I_\zeta\omega_\zeta^2.$$

Using Eqs. (159) of the preceding article, we have then

$$2T = H_\xi\omega_\xi + H_\eta\omega_\eta + H_\zeta\omega_\zeta = H\Omega\cos\alpha. \tag{167}$$

Since T, H, and Ω are all positive quantities, designating the kinetic energy, the angular momentum, and the resultant angular velocity of the gyroscope, respectively, $\cos\alpha$ must be positive. Hence α is an acute angle.

To visualize regular precession of a gyroscope, we shall use the notion of a *body cone* rolling on a *space cone* (see page 329). We shall agree also to take the immovable z-axis in the direction of the vector ω_1, representing the velocity of precession, and the movable ζ-axis of the gyroscope in the direction of the vector ω. One case of regular precession is shown in Fig. 253a. The body cone touches the space cone from the outside, and the regular precession is visualized by rolling of the cone AOB on the immovable cone AOC. The instantaneous axis of the gyroscope coincides at any instant with

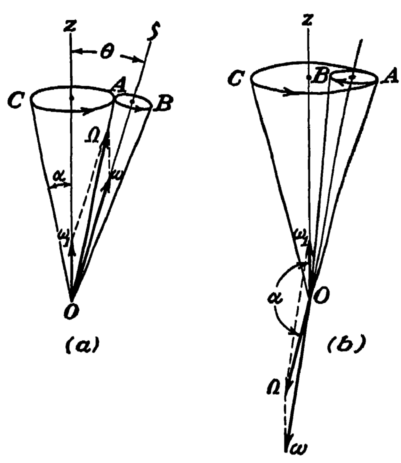

FIG. 253.

the line of tangency OA, and the resultant angular velocity Ω is directed along that line. The case of the body cone rolling inside the space cone (Fig. 253b) is dynamically impossible, since the condition of an acute angle α between ω_1 and Ω is not satisfied in this case.

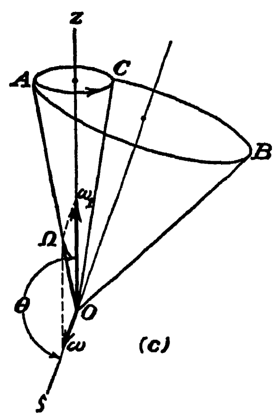

FIG. 254.

Another possible case of regular precession is shown in Fig. 254. Here again the body cone AOB touches the space cone AOC from the outside. The line of tangency OA is the instantaneous axis of rotation, and the resultant angular velocity Ω is directed along this axis. The angular velocity ω_1 coincides with the z-axis so that the direction of ω and the ζ-axis will be as shown. Comparing Figs. 253a and 254, we see that in the first case, the angle θ is acute while, in the second case, it is obtuse. For an observer looking from z toward O in Fig. 253a, both rotations are counterclockwise, while in Fig. 254 the rotation corresponding to ω_1 is counterclockwise, and the rotation corresponding to ω is clockwise. The first kind of motion is called *direct or progressive precession*, and the second is called *retrograde precession.*

To decide what kind of motion we shall have in a given case, we have to consider Eq. (165). Assume first that the gyroscope has a form elongated in the direction of its axis. Then the axial moment of inertia I is smaller than the equatorial moment of inertia I_1, and the quantity $I_1 - I$ is positive. The angular velocities ω_1 and ω, always having the directions of the z- and ζ-axes, are also positive, and we conclude from Eq. (165) that $\cos\theta$ is positive; hence θ is an acute angle. Thus a gyroscope of elongated form will perform direct precession. If the gyroscope has a flattened form like those shown in Fig. 244, the quantity $I_1 - I$ will be negative and we conclude from Eq. (165) that θ is an obtuse angle. Such a gyroscope will perform retrograde precession.

From the above discussion of the solution of Eqs. (163), we conclude that in the absence of external forces, the motion of a symmetrical gyroscope is always a regular precession. To bring a gyroscope to such motion, we can apply an impulse. Since the center of gravity of the gyroscope is fixed, any impulsive force will produce an equal and opposite reaction and we have, in effect, an impulsive couple applied to the body. If a gyroscope performing regular precession suffers an infinitesimal impulse by a couple the plane of which is perpendicular to the angular momentum H, such an impulse will produce only an infinitesimal change in the magnitude of the angular momentum without changing its direc-

tion. The direction of the instantaneous axis will also remain unchanged, but the angular velocity Ω about this axis and both of its components ω and ω_1 will obtain certain increments. If ΔH is the infinitesimal increment of the angular momentum, equal to the produced impulse $\Delta M'$, we find, from Eq. (166),

$$\Delta\omega_1 = \frac{\Delta H}{I_1} = \frac{\Delta M'}{I_1}. \tag{d}$$

Then from Eq. (165), we obtain

$$\Delta\omega = \Delta\omega_1 \frac{(I_1 - I)\cos\theta}{I} = \Delta M' \frac{(I_1 - I)\cos\theta}{II_1}. \tag{e}$$

If a couple M acts continuously in the plane perpendicular to the vector H, we can divide the time into infinitesimal intervals and assume that during each of those intervals, an infinitesimal impulse $dM' = M\,dt$ is communicated to the gyroscope. Equations (*d*) and (*e*) then give

$$\left.\begin{aligned} \frac{d\omega_1}{dt} &= \frac{M}{I_1}, \\ \frac{d\omega}{dt} &= \frac{M(I_1 - I)\cos\theta}{II_1}. \end{aligned}\right\} \tag{168}$$

If M is known as a function of time, these equations can be readily integrated and we shall find the angular velocities ω and ω_1. The angle θ remains constant during this motion and equal to its initial value.

Let us consider now the effect produced on the regular precession of a gyroscope by an impulsive couple the plane of which passes through the vector H, representing the angular momentum of the gyroscope. Assume first that the gyroscope rotates about its axis of symmetry, as about a fixed axis, and that the vector $H = I\omega$ represents its angular momentum (Fig. 255). If an impulsive force acts on the gyroscope's axis in the plane perpendicular to the plane of the figure, this force, together with the reaction at the fixed point O, forms an impulsive couple M, the impulse $M\ \Delta t$ of which is shown in the figure by the vector M'. This

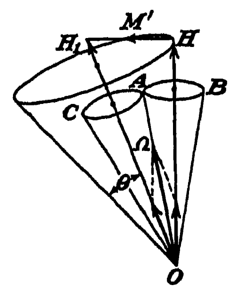

FIG. 255.

vector is equal to the change that the impulse produces in the magnitude of the angular momentum of the gyroscope, so that this momentum, after impact, will be represented by the vector H_1. The gyroscope, after impact, will perform regular precession, the vector H_1 will remain immovable in space, and the axis of the gyroscope will describe the *precession cone* about that vector. This precession is visualized in Fig. 255 by a body cone AOB rolling on a space cone AOC. It is

seen that at the instant of impact, the impulsive force, acting perpendicularly to the plane of the figure, produces a displacement of the axis of rotation from the position OH to the position OA. This displacement takes place in the direction perpendicular to the direction of the impulsive force. At the same time, the axis of the gyroscope begins to describe the precession cone with the apex angle 2θ. The angle θ between the axis of the gyroscope and the vector H_1 is determined from the equation

$$\theta = \arctan \frac{M'}{H}. \qquad (f)$$

We see that if a high angular velocity is initially communicated to the gyroscope so that H is a large quantity, the angle θ will be very small. Thus a high-speed gyroscope has the tendency to preserve the direction of its axis and resists the actions of external impulses. This property of the gyroscope is utilized in many technical applications, some of which will be discussed later.

Knowing the effect produced on the motion of a gyroscope by an impulsive couple, we can discuss now the action of a continuously applied couple M which can be treated as a series of infinitesimal impulses $M\,\Delta t$ each of which produces an effect similar to that shown in Fig. 255. The most important case of this kind is the action of the gravity force when the center of gravity C of the gyroscope does not coincide with the point of support O as shown in Fig. 256. We assume that the gyroscope is rotating with high speed about its own axis, so that with good accuracy we can take the angular momentum H coincident with the axis of the gyroscope which has an angle of inclination δ to the vertical z-axis. For a constant angle δ, the gravity force W, together with the reaction at O, forms a couple M of constant magnitude acting in the plane DOH. From the principle of angular momentum, we conclude that during motion, the vector representing this couple is equal to the velocity v of point H, the end of the vector representing the angular momentum. This velocity will be perpendicular to DH, and point H will describe a circle of radius $\overline{DH} = H \sin \delta$. Denoting by ω_1 the angular velocity with which the plane DOH rotates about the z-axis and equating the velocity v to the moment M of external forces, we obtain

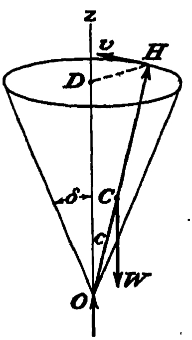

FIG. 256.

$$H \sin \delta \cdot \omega_1 = M,$$

from which

$$\omega_1 = \frac{M}{H \sin \delta} = \frac{Wc}{I\omega}. \qquad (169)$$

We see that owing to the action of the gravity force, the axis of the gyroscope describes a cone with constant angular velocity ω_1, and we have a motion similar to the regular precession discussed before. However, there is one important difference between the two motions. The axis of the gyroscope in this case does not coincide exactly with the vector H; and owing to this fact, the motion is much more complicated than it looks. There is a *secondary precession* of the axis of the gyroscope around the vector H like that visualized in Fig. 255. To represent this complicated motion, we can imagine a spherical surface described around the fixed point O. Then the end F of the axis of the gyroscope will describe a complicated curve on this sphere, one example of which is shown in Fig. 257. In this figure, δ denotes the angle between the vector H and the immovable z-axis and θ denotes the small angle of the secondary precession. It can be seen that owing to this secondary precession, the moment of the gravity force does not remain constant and the motion is not so simple as it was assumed at the beginning. This kind of motion is called *pseudoregular precession* and will be discussed in more detail at the end of the next article.

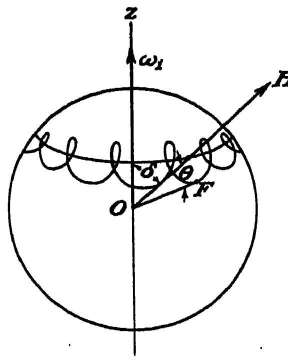

FIG. 257.

46. Stability of Free Motion of a Gyroscope.—Let us now consider motion of a symmetrical gyroscope of the type shown in Fig. 244*b*. In writing equations of motion for this case, we shall use the modified Euler's equations (162) and take for ξ', η', ζ', the axes OM, ON, and $O\zeta$, respectively, as shown in Fig. 252. The axes $O\zeta$ and ON we take to coincide with OO and O_1O_1 in Fig. 244*b* and representing, respectively, the axis of the gyroscope and the axis of rotation of the inner frame NN. Let H_m, H_n, H_ζ be the components of the angular momentum H and ω_m', ω_n', ω_ζ' the components of the angular velocity of the trihedron $OMN\zeta$ in Fig. 252. In calculating these components of the angular velocity, we observe that the position of the trihedron $OMN\zeta$ is completely defined by the angles ψ and θ. The corresponding components of the angular velocity in the directions of Oz and ON are $\dot{\psi}$ and $\dot{\theta}$, respectively. Projecting these components on the moving axes OM, ON, $O\zeta$, we obtain

$$\omega_m' = -\dot{\psi}\sin\theta, \qquad \omega_n' = \dot{\theta}, \qquad \omega_\zeta' = \dot{\psi}\cos\theta. \tag{a}$$

The components ω_m, ω_n, ω_ζ of the angular velocity of the body of the gyroscope are obtained by considering, in addition to the components (*a*), the angular velocity $\dot{\phi}$ that the gyroscope has with respect to the trihedron $OMN\zeta$. This gives

$$\omega_m = -\dot{\psi}\sin\theta, \qquad \omega_n = \dot{\theta}, \qquad \omega_\zeta = \dot{\phi} + \dot{\psi}\cos\theta. \tag{b}$$

Using these values of the components of the angular velocity of the gyroscope and observing that for a symmetrical gyroscope, ON and OM are always principal axes, we obtain[1]

[1] We neglect the masses of the frames and consider only the mass of the gyroscope.

$$H_m = -I_1\dot{\psi}\sin\theta, \qquad H_n = I_1\dot{\theta}, \qquad H_\zeta = I(\dot{\phi} + \dot{\psi}\cos\theta). \tag{c}$$

Substituting these components of the angular momentum, together with the components (*a*) of the angular velocity ω', into Eqs. (162), we obtain for equations of motion

$$\left.\begin{aligned} -I_1\frac{d}{dt}(\dot{\psi}\sin\theta) + I\dot{\theta}(\dot{\phi} + \dot{\psi}\cos\theta) - I_1\dot{\psi}\dot{\theta}\cos\theta &= M_m, \\ I_1\ddot{\theta} - I_1\dot{\psi}^2\sin\theta\cos\theta + I\dot{\psi}\sin\theta(\dot{\phi} + \dot{\psi}\cos\theta) &= M_n, \\ I\frac{d}{dt}(\dot{\phi} + \dot{\psi}\cos\theta) &= M_\zeta, \end{aligned}\right\} \tag{d}$$

in which M_m, M_n, and M_ζ are the moments of external forces acting on the gyroscope with respect to the axes OM, ON, and $O\zeta$, respectively.

In our further discussion, we assume that the gyroscope rotates with respect to its own axis with a high angular velocity, so that $\dot{\phi}$ is very large in comparison with $\dot{\psi}$ and $\dot{\theta}$. Neglecting then in Eqs. (*d*) the terms containing as factors $\dot{\psi}$ or $\dot{\theta}$ in comparison with terms containing $\dot{\phi}$, we obtain

$$\left.\begin{aligned} -I_1\ddot{\psi}\sin\theta + I\dot{\theta}\dot{\phi} &= M_m, \\ I_1\ddot{\theta} + I\dot{\phi}\dot{\psi}\sin\theta &= M_n, \\ I\frac{d}{dt}(\dot{\phi} + \dot{\psi}\cos\theta) &= M_\zeta, \end{aligned}\right\} \tag{e}$$

Assume now that there are no external forces[1] acting on the gyroscope; then from the third of Eqs. (*e*) we conclude that $\dot{\phi} + \dot{\psi}\cos\theta$ is constant and, with sufficient accuracy, we can neglect $\dot{\psi}\cos\theta$ in comparison with $\dot{\phi}$ and assume $\dot{\phi} = \omega$. Accordingly, the first two of Eqs. (*e*) give

$$\left.\begin{aligned} -I_1\ddot{\psi}\sin\theta + I\omega\dot{\theta} &= 0. \\ I_1\ddot{\theta} + I\omega\dot{\psi}\sin\theta &= 0. \end{aligned}\right\} \tag{f}$$

In the solution of these equations, we assume that the axis of the gyroscope is placed at right angles to the vertical z-axis so that $\theta = \pi/2$ and also that $\psi = 0$ when $t = 0$. Giving to the gyroscope by an impulse some small initial velocities $\dot{\theta}_0$ and $\dot{\psi}_0$, we may investigate, by Eqs. (*f*), the motion that ensues. If the corresponding displacements have the tendency to remain always small, we shall have the proof that a rapidly rotating gyroscope is in a condition of stable motion. Proceeding as explained in Art. 41 and assuming that $\sin\theta$ differs from unity only by a small quantity, we obtain, from Eqs. (*f*),

$$\left.\begin{aligned} -I_1\ddot{\psi} + I\omega\dot{\theta} &= 0, \\ I_1\ddot{\theta} + I\omega\dot{\psi} &= 0. \end{aligned}\right\} \tag{g}$$

Eliminating θ, we have

$$I_1^2\dddot{\psi} + I^2\omega^2\dot{\psi} = 0. \tag{h}$$

Introducing the notation

$$p^2 = \frac{I^2\omega^2}{I_1^2}, \tag{i}$$

we can write the solution of Eq. (*h*) in the following form:

$$\psi = C_1 + C_2\cos pt + C_3\sin pt,$$

[1] This means that the intersection point of the three axes of rotation in Fig. 244*b* coincides with the center of gravity of the gyroscope. Taking M_ζ equal to zero, we also neglect friction in the bearings.

where C_1, C_2, C_3 are constants of integration. Differentiating twice with respect to time and substituting into the first of Eqs. (*g*), we obtain

$$\dot{\theta} = -\frac{I_1p^2}{I\omega}(C_2\cos pt + C_3\sin pt).$$

Then, by integration, we find

$$\theta = -\frac{I_1p}{I\omega}(C_2\sin pt - C_3\cos pt) + C_4 = -C_2\sin pt + C_3\cos pt + C_4.$$

To satisfy the assumed initial conditions

$$\theta = \frac{\pi}{2}, \qquad \psi = 0, \qquad \dot{\theta} = \dot{\theta}_0, \qquad \dot{\psi} = \dot{\psi}_0, \qquad \text{when } t = 0,$$

we must take

$$C_1 + C_2 = 0, \qquad C_4 + C_3 = \frac{\pi}{2}, \qquad C_3p = \dot{\psi}_0, \qquad -C_2p = \dot{\theta}_0,$$

and the final solution becomes

$$\left.\begin{aligned} \psi &= \frac{\dot{\theta}_0}{p}(1 - \cos pt) + \frac{\dot{\psi}_0}{p}\sin pt, \\ \theta &= \frac{\pi}{2} - \frac{\dot{\psi}_0}{p}(1 - \cos pt) + \frac{\dot{\theta}_0}{p}\sin pt. \end{aligned}\right\} \qquad (j)$$

We see that if a small impulse is given to the axis of the rapidly rotating gyroscope, the axis begins to perform harmonic oscillations of small amplitude about the position defined by the angles

$$\psi = \frac{\dot{\theta}_0}{p}, \qquad \theta = \frac{\pi}{2} - \frac{\dot{\psi}_0}{p}.$$

The frequency of these oscillations, as we see from expression (*i*), is very large, so that in experimenting with the gyroscope it is difficult to notice them. We assumed in our derivation that $\theta = \pi/2$, but the proof can be extended without difficulty to any other value of θ, and it can be concluded that a rapidly rotating gyroscope, supported as shown in Fig. 244*b*, stably retains the direction of its axis.

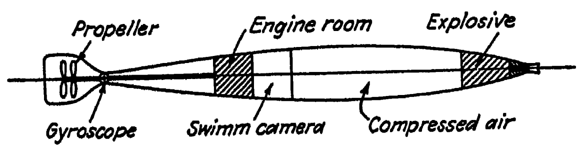

FIG. 258.

It must be noted that this stability characteristic vanishes if one of the three degrees of freedom of the gyroscope is removed by some constraint. For example, if we fix the axis O_2O_2 of the outer frame PP in Fig. 244*b* and give some initial angular velocity to the inner frame NN, this frame will continue to rotate with this velocity about the axis O_1O_1 and the direction of the axis of the gyroscope is no longer stably retained. The moment of external forces corresponding to variation of the angular momentum of the gyroscope in this case is furnished by the reactions on the fixed axis O_2O_2.

The property of a gyroscope to retain stably the direction of its axis of rotation is utilized in various technical applications. We shall discuss here the use of this property in the control of the straight-line motion of the Whitehead torpedo, shown

in Fig. 258. This torpedo is about 16 ft. long and 18 in. in maximum diameter and is propelled by a compressed-air engine at a speed of about 50 f.p.s.[1] To control the motion of the torpedo in a horizontal plane, the gyroscope shown in Fig. 259 is used. The first device of this kind was constructed by the Austrian engineer Obry.[2] His gyroscope weighed about 1.8 lb. and rotated at about 10,000 r.p.m. The frame MM of the gyroscope is attached to the body of the torpedo, and the axis OO of the gyroscope at the instant of shooting coincides with the longitudinal axis of the torpedo. If, in its motion in a horizontal plane, the torpedo deviates from a straight line, the gyroscope retains the direction of its axis OO and some rotation of the outer ring about its vertical axis O_1O_1 takes place. This rotation brings into action a device controlling the rudder of the torpedo. To assure satisfactory performance of the gyroscope, friction forces, entirely neglected in the theoretical discussion of this article, must be reduced to a minimum.

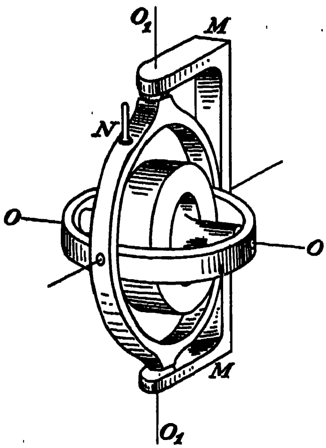

FIG. 259.

Consider now the case in which a constant moment M_n acts on the gyroscope with respect to the moving axis ON coinciding with O_1O_1 in Fig. 244*b*. Then, instead of Eqs. (*g*), we write

$$\left.\begin{aligned} -I_1\ddot{\psi} + I\omega\dot{\theta} &= 0,\\ I_1\ddot{\theta} + I\omega\dot{\psi} &= M_n. \end{aligned}\right\} \tag{k}$$

Proceeding as before, we find

$$\dddot{\psi} + p^2\dot{\psi} = \frac{p^2}{I\omega} M_n.$$

The general solution of this equation is

$$\dot{\psi} = C_1 + C_2\cos pt + C_3\sin pt + \frac{M_n}{I\omega}t.$$

Substituting in the first of Eqs. (*k*), we find

$$\dot{\theta} = \frac{I_1}{I\omega}\ddot{\psi} = -p(C_2\cos pt + C_3\sin pt)$$

and, by integration,

$$\theta = -C_2\sin pt + C_3\cos pt + C_4.$$

The constants of integration $C_1 \ldots C_4$ have to be determined in each particular case from the initial conditions of motion. Assume, for example, that for $t = 0$,

$$\psi = 0, \qquad \theta = \frac{\pi}{2}, \qquad \dot{\psi} = \dot{\psi}_0, \qquad \dot{\theta} = 0.$$

Then

$$C_1 + C_2 = 0, \qquad C_3p + \frac{M_n}{I\omega} = \dot{\psi}_0, \qquad C_3 + C_4 = \frac{\pi}{2}, \qquad C_2 = 0,$$

and we find

$$C_1 = C_2 = 0, \qquad C_3 = \frac{1}{p}\left(\dot{\psi}_0 - \frac{M_n}{I\omega}\right), \qquad C_4 = \frac{\pi}{2} - \frac{1}{p}\left(\dot{\psi}_0 - \frac{M_n}{I\omega}\right).$$

[1] All these data refer to a torpedo of about the year 1900.

[2] For a description of this device, see paper by W. J. Sears, *Engineering*, vol. 66, p. 89, 1898.

The angles ψ and θ then are

$$\left.\begin{aligned}\psi &= \frac{1}{p}\left(\dot\psi_0 - \frac{M_n}{I\omega}\right)\sin pt + \frac{M_n t}{I\omega},\\ \theta &= \frac{1}{p}\left(\dot\psi_0 - \frac{M_n}{I\omega}\right)\cos pt + \frac{\pi}{2} - \frac{1}{p}\left(\dot\psi_0 - \frac{M_n}{I\omega}\right).\end{aligned}\right\} \tag{l}$$

The terms

$$\frac{\pi}{2} - \frac{1}{p}\left(\dot\psi_0 - \frac{M_n}{I\omega}\right) \qquad \text{and} \qquad \frac{M_n t}{I\omega}$$

indicate that owing to the initial velocity $\dot\psi_0$, the axis of the gyroscope is slightly displaced from the direction perpendicular to the z-axis and rotates about that axis with uniform angular velocity[1]

$$\omega_1 = \frac{M_n}{I\omega}. \tag{m}$$

On this *regular precession*, small high-frequency vibrations represented by the trigonometric terms in Eqs. (l) are superimposed. The resultant motion is the *pseudoregular precession* illustrated in Fig. 257, page 341. The various shapes of the curves that will be described by the end of the axis of the gyroscope on the spherical surface in Fig. 257 will depend on the values of $\dot\psi_0$ and the quantity $M_n/I\omega$. The constant moment M_n assumed in our discussion may be produced by the gravity force if the center of gravity of the gyroscope does not coincide with the point of support, and such a gyroscope usually performs *pseudoregular precession*. If the initial velocity $\dot\psi_0$ communicated to the gyroscope is selected so that

$$\dot\psi_0 - \frac{M_n}{I\omega} = 0, \tag{n}$$

the trigonometric terms in Eqs. (l) vanish and the gyroscope will perform regular precession. We see that in the case of an external moment M_n, the regular precession is only a particular case of motion which can be produced by a proper selection of the initial conditions so as to satisfy condition (n).

47. Gyroscopic Moment of a Symmetrical Gyroscope.—In the preceding articles, several simple cases of the solution of Euler's equations (161) and (162) were discussed and gyroscopic motion produced by given forces was investigated. Let us consider now the reverse problem and assume that the motion of a gyroscope is known and that it is required to find the external forces which must be applied to the gyroscope to produce the assumed motion. This problem is comparatively simple. Knowing the motion, we can calculate without difficulty the left-hand side of Eq. (161) or (162) and, in this way, obtain the components of the resultant external moment acting on the gyroscope. The moment of the same magnitude but of opposite sign will represent the action of the gyroscope on its bearings, and this is called the *gyroscopic moment*. There are many practical problems where the calculation of this gyroscopic moment is required, and we shall consider here several such examples.

[1] This coincides with the result previously obtained; see Eq. (169), p. 340.

Let us take first the case of regular precession. A symmetrical gyroscope rotates with constant angular velocity ω about its axis, while this axis describes, with uniform angular velocity ω_1, a precession cone having the angle 2θ at its apex (Fig. 260). We take equations of motion (162) and use again, as the system of moving axes, the trihedron $OMN\zeta$. In calculating the projections of the angular velocity of the trihedron on these axes, we have to consider only the angular velocity $\dot{\psi} = \omega_1$ about the immovable z-axis. Projecting this velocity, we obtain

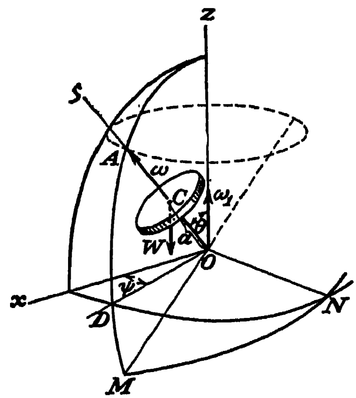

FIG. 260.

$$\left.\begin{aligned} \omega_m' &= -\omega_1 \sin\theta, \\ \omega_n' &= 0, \\ \omega_\zeta' &= \omega_1 \cos\theta. \end{aligned}\right\} \tag{a}$$

In calculating the components of the angular velocity of the gyroscope, we have to consider also the angular velocity ω with respect to the trihedron $OMN\zeta$, which gives

$$\omega_m = -\omega_1 \sin\theta, \qquad \omega_n = 0, \qquad \omega_\zeta = \omega + \omega_1 \cos\theta. \tag{b}$$

The corresponding values of the components of the angular momentum of the gyroscope are

$$H_m = -I_1\omega_1 \sin\theta, \qquad H_n = 0, \qquad H_\zeta = I(\omega + \omega_1 \cos\theta). \tag{c}$$

Substituting expressions (a) and (c) into Eqs. (162), we find the components of the resultant moment acting on the gyroscope to be

$$\left.\begin{aligned} M_m &= 0, \\ M_n &= I\omega\omega_1 \sin\theta + (I - I_1)\omega_1^2 \sin\theta\cos\theta, \\ M_\zeta &= 0. \end{aligned}\right\} \tag{d}$$

We conclude that the gyroscopic moment in this case acts in the $zO\zeta$-plane and that its magnitude is

$$\mathfrak{M}_n = -M_n = -[I\omega + (I - I_1)\omega_1 \cos\theta]\omega_1 \sin\theta. \tag{170}$$

In practical applications, the angular velocity ω of the gyroscope about its own axis is usually very large and the first term in the brackets, which is always positive, is large in comparison with the second term. This indicates that the gyroscopic moment (170) is negative. It always acts so as to bring the vector ω into coincidence with the vector ω_1 in Fig. 260.

If the angular velocity ω of the gyroscope is such that

$$I\omega + (I - I_1)\omega_1 \cos\theta = 0, \tag{e}$$

the condition (164) on page 336 is satisfied, the gyroscopic moment vanishes, and the gyroscope, in the absence of external forces, performs the regular precession already discussed in Art. 45.

In a general case of motion of the gyroscope, the position of the trihedron $OMN\zeta$ is defined by the angles ψ and θ in Fig. 260. Its angular velocity has the components $\dot{\psi}$ and $\dot{\theta}$ along the z- and ON-axes, respectively, and we find

$$\omega_m' = -\dot{\psi} \sin \theta, \qquad \omega_n' = \dot{\theta}, \qquad \omega_\zeta' = \dot{\psi} \cos \theta. \tag{f}$$

The corresponding components of the angular velocity of the gyroscope are

$$\omega_m = -\dot{\psi} \sin \theta, \qquad \omega_n = \dot{\theta}, \qquad \omega_\zeta = \dot{\phi} + \dot{\psi} \cos \theta, \tag{g}$$

and the components of its angular momentum are

$$H_m = -I_1\dot{\psi} \sin \theta, \qquad H_n = I_1\dot{\theta}, \qquad H_\zeta = I(\dot{\phi} + \dot{\psi} \cos \theta). \tag{h}$$

Equations (162) then give

$$\left.\begin{aligned} M_m &= -I_1 \frac{d}{dt} (\dot{\psi} \sin \theta) + \dot{\theta} I(\dot{\phi} + \dot{\psi} \cos \theta) - I_1\dot{\theta}\dot{\psi} \cos \theta, \\ M_n &= I_1\ddot{\theta} + \dot{\psi} \sin \theta\,[I\dot{\phi} + (I - I_1)\,\dot{\psi} \cos \theta], \\ M_\zeta &= I \frac{d}{dt} (\dot{\phi} + \dot{\psi} \cos \theta). \end{aligned}\right\} \tag{171}$$

When the motion of the gyroscope is given, the angles ϕ, ψ, θ are all known functions of t. Substituting them into the right-hand sides of Eqs. (171), we obtain the components M_m, M_n, M_ζ of the resultant moment acting on the gyroscope. The components of the gyroscopic moment will be $\mathfrak{M}_m = -M_m$, $\mathfrak{M}_n = -M_n$ and $\mathfrak{M}_\zeta = -M_\zeta$.

In the case of a gyroscope having a very high angular velocity about its own axis, we can neglect the components H_m and H_n of the angular momentum in comparison with H_ζ and also $\dot{\psi}$ and $\dot{\theta}$ in comparison with $\dot{\phi}$. Then Eqs. (171) give

$$\left.\begin{aligned} M_m &= I\dot{\theta}\dot{\phi} = -\mathfrak{M}_m, \\ M_n &= I\dot{\psi}\dot{\phi} \sin \theta = -\mathfrak{M}_n, \\ M_\zeta &= I\ddot{\phi} = -\mathfrak{M}_\zeta. \end{aligned}\right\} \tag{172}$$

These equations coincide with those usually given in the elementary theory of gyroscopes, where it is assumed that the angular momentum of the gyroscope is represented by a vector directed along the axis of spin.

As an example of calculation of the gyroscopic moment (172), let us investigate the reactions at the bearings of a uniformly rotating shaft carrying a circular disk the axis of symmetry of which makes an angle θ

with the axis of the shaft (Fig. 261). Considering the disk as a gyroscope, the angular velocity of which with respect to its own axis vanishes and whose velocity of precession ω_1 is equal to the velocity of the shaft, we obtain from Eq. (170) the gyroscopic moment

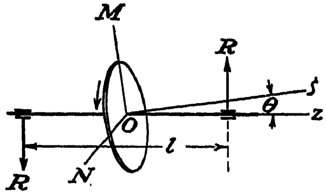

FIG. 261.

$$\mathfrak{M}_n = -(I - I_1)\omega_1^2 \sin\theta\cos\theta.$$

Since, for a disk, $I > I_1$, the gyroscopic moment is negative and the reactions acting on the shaft are directed as shown in the figure. The magnitude R of these reactions is found from the equation

$$Rl = (I - I_1)\omega_1^2 \sin\theta\cos\theta. \qquad (i)$$

We see that in the case of a large velocity ω_1 of the shaft, the reactions can become very large; and to eliminate them, great care must be used in fixing the plane of the disk at right angles to the axis of the shaft. This example represents the case of dynamic unbalance which is so important in high-speed machines.[1]

As a second example, let us discuss the gyroscopic action on a pair of locomotive wheels during the negotiation of a curve in the track (Fig.

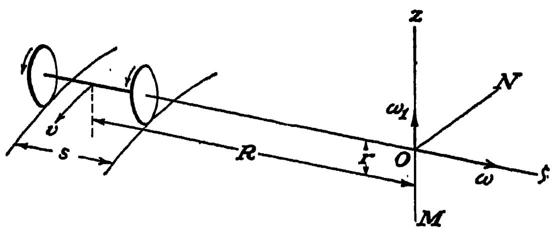

FIG. 262.

262). Considering the locomotive axle as a symmetrical gyroscope performing regular precession and using notations v, r, and R for the speed of the locomotive, the radius of the wheel, and the radius of the curve, respectively, we find, for the velocity of precession ω_1 and for the angular velocity of the wheels about their axis, the values

$$\omega_1 = \frac{v}{R}, \qquad \omega = \frac{v}{r}. \qquad (j)$$

Substituting these values in expression (170) and putting $\theta = \pi/2$, we find that the gyroscopic moment acting in the rotating $zO\zeta$-plane has

[1] See the authors' "Engineering Mechanics," 2d ed., p. 421.

the magnitude

$$\mathfrak{M}_n = -I\omega\omega_1 = -\frac{Iv^2}{Rr}. \tag{k}$$

This is the moment about the ON-axis in Fig. 262, and the minus sign indicates that the direction of the moment is such as to increase the rail pressure on the outer wheel and diminish that on the inner wheel. This gyroscopic action is in the same direction as the action of the centrifugal force applied at the center of gravity of the locomotive, and both moments are proportional to v^2. In addition, the gyroscopic moment is proportional to I, so that it becomes of greater importance, for example, in the case of electric locomotives having the motors directly on the axles. Calculations made for one such locomotive showed that the gyroscopic moment was about 6 per cent of the moment due to centrifugal force.[1]

In the preceding discussion, the railroad track was assumed to be in a horizontal plane. Usually, to counteract the centrifugal force, some superelevation is given to the outer rail on the curve so that during transition from tangent to curve, the axle has some rotation in the vertical plane. Assuming a uniform increase in elevation during this transition, the corresponding angular velocity is

$$\dot{\theta} = \frac{h}{s}\frac{v}{l},$$

where h is the superelevation, s the track gauge, and l the length of the transition curve. The corresponding rate of change of angular momentum, indicated in Fig. 263, is

$$I\omega\dot{\theta} = \frac{Iv^2h}{rsl}. \tag{l}$$

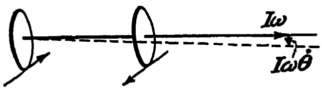

FIG. 263.

This means that during transition, a horizontal couple, furnished by friction forces, must act on the wheels as shown. The ratio of the two moments (l) and (k) is

$$\frac{h}{s}\frac{R}{l}.$$

This ratio may sometimes become larger than unity, and high-speed traction experiments have shown that for such speeds as 140 m.p.h., it is important to increase the length l of the transition curve.

An external moment of the type (l) can also be produced by a deviation of the wheels from a true circular shape or by unequal deflections of the two rails. In all these cases, a friction moment like that shown in Fig. 263 is produced; its magnitude increases in proportion to the square

[1] See *Z. Ver. deut. Ing.*, vol. 48, p. 949, 1904.

of the locomotive speed, and at high speeds it may become of practical importance.

As a third example, let us consider the gyroscopic moment produced by a turbine on a pitching ship if the axis of the turbine is parallel to the longitudinal axis of the ship. Let ω be the angular velocity and $I\omega$ the angular momentum of the turbine. The pitching of the ship is represented by the equation

$$\theta = a \sin \frac{2\pi t}{\tau},$$

where a is the amplitude of oscillation and τ is its period. The rate of change of the angular momentum of the turbine then is

$$I\omega\dot{\theta} = I\omega a \frac{2\pi}{\tau} \cos \frac{2\pi t}{\tau}.$$

Its maximum value, equal to the maximum value of the gyroscopic moment acting in the horizontal plane, is

$$I\omega a \frac{2\pi}{\tau}.$$

Assume, for example, that the turbine is making 300 r.p.m. and that the weight of the rotor, shaft, and propeller is 80 tons with a radius of gyration of 4 ft. Then assuming that the period of pitching $\tau = 12$ sec. and the angular amplitude $a = \frac{1}{8}$ rad., the maximum value of the gyroscopic moment is 82.2 ton-ft. Although this moment seems large, its action on the motion of a large ship is negligible. However, it should be considered in discussing pressure of the turbine shaft on its bearings.

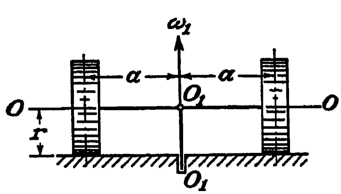

FIG. 264.

As a last example, let us consider the gyroscopic moment of the rollers of a rolling mill as shown in Fig. 264. If ω_1 is the angular velocity of the mill about the vertical axis O_1O_1, the angular velocity of each roller about its own axis OO is

$$\omega = \frac{\omega_1 a}{r}$$

and the magnitude of the gyroscopic moment, from Eq. (170), is

$$\mathfrak{M} = I\omega\omega_1 = I\frac{\omega_1^2 a}{r}.$$

Its direction is such as to increase the pressure of the rollers on the supporting horizontal plane. As a result of this gyroscopic action, tension

will be produded in the vertical driving axle O_1O_1, which must be considered in its design.

Let us consider now the case where the axis of the roller is inclined to the vertical axis by an angle θ, as shown in Fig. 265. When the roller is at rest, the pressure at A is obtained by writing an equation of moments with respect to point O and equating the moment of the gravity force W to the moment of the reaction at A. When the roller rotates with angular velocity ω_1 about the vertical axis, the angular velocity ω about its own axis is found from the condition that OA is the instantaneous axis of rotation. Hence

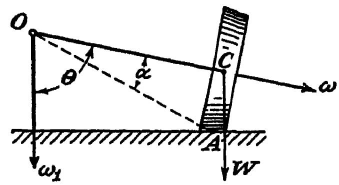

FIG. 265.

$$\omega = \omega_1 \frac{\sin(\theta - \alpha)}{\sin \alpha}.$$

Substituting into Eq. (170), we obtain the gyroscopic moment that must be combined with the moment of the gravity force about point O in calculating the pressure of the roller at A. The quantity I in Eq. (170) is the moment of inertia of the roller about its axis, while I_1 is the moment of inertia about an axis perpendicular to the axis of the roller and passing through the immovable point O.

48. Gyroscopic Moment of an Unsymmetrical Gyroscope.—In our preceding calculation of the gyroscopic moment [Eq. (170)], we assumed a symmetrical gyroscope, but there are cases of practical interest in which gyroscopes are not symmetrical. A practically important example of such a gyroscope is represented by a two-blade airplane propeller. The gyroscopic moment in this case represents the action on the bearings of the propeller shaft due to any deviation of the airplane from a straight-line course. In deriving the gyroscopic moment for an unsymmetrical gyroscope moving about an immovable point O (Fig. 266), we proceed as before and take the axis of the gyroscope as the ζ-axis and select the axes OM and ON as shown in the figure. Then denoting by ω_m', ω_n', ω_ζ' the components of the angular velocity of the trihedron $OMN\zeta$ and by H_m, H_n, H_ζ the corresponding components of the angular momentum of the gyroscope, we obtain, by using Eqs. (162), the following expressions for the components of the required gyroscopic moment:

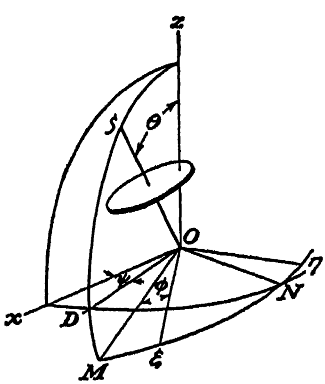

FIG. 266.

$$\left.\begin{aligned}\mathfrak{M}_m &= -M_m = -\left(\frac{dH_m}{dt} + \omega_n' H_\zeta - \omega_\zeta' H_n\right),\\ \mathfrak{M}_n &= -M_n = -\left(\frac{dH_n}{dt} + \omega_\zeta' H_m - \omega_m' H_\zeta\right),\\ \mathfrak{M}_\zeta &= -M_\zeta = -\left(\frac{dH_\zeta}{dt} + \omega_m' H_n - \omega_n' H_m\right)\end{aligned}\right\} \qquad (a)$$

The complication of further calculation comes from the fact that in the case of an unsymmetrical gyroscope, the axes OM and ON are no longer principal axes of inertia and we cannot use the simple expressions (159) in calculating the components H_m and H_n of the angular momentum of the gyroscope. To derive the expressions for these two components, we use first the axes $O\xi$ and $O\eta$ in Fig. 266, attached to the gyroscope and directed along its principal axes through the fixed point O. For these axes, we can write

$$H_\xi = I_\xi \omega_\xi, \qquad H_\eta = I_\eta \omega_\eta. \qquad (b)$$

The components ω_ξ and ω_η of the angular velocity of the gyroscope will now be obtained by projecting on the ξ- and η-axes the components ω_m and ω_n as given by expressions (g) of the preceding article. In this way, we obtain

$$\left.\begin{aligned}\omega_\xi &= \omega_m \cos\phi + \omega_n \sin\phi = -\dot\psi \sin\theta\cos\phi + \dot\theta\sin\phi,\\ \omega_\eta &= -\omega_m \sin\phi + \omega_n\cos\phi = \dot\psi\sin\theta\sin\phi + \dot\theta\cos\phi.\end{aligned}\right\} \qquad (c)$$

Substituting into Eqs. (b), we obtain

$$\left.\begin{aligned}H_\xi &= I_\xi(-\dot\psi\sin\theta\cos\phi + \dot\theta\sin\phi),\\ H_\eta &= I_\eta(\dot\psi\sin\theta\sin\phi + \dot\theta\cos\phi).\end{aligned}\right\} \qquad (d)$$

The required components H_m and H_n of the angular momentum are now obtained by projecting H_ξ and H_η on the OM- and ON-axes, which gives

$$\left.\begin{aligned}H_m &= I_\xi(-\dot\psi\sin\theta\cos\phi + \dot\theta\sin\phi)\cos\phi\\ &\qquad - I_\eta(\dot\psi\sin\theta\sin\phi + \dot\theta\cos\phi)\sin\phi,\\ H_n &= I_\xi(-\dot\psi\sin\theta\cos\phi + \dot\theta\sin\phi)\sin\phi\\ &\qquad + I_\eta(\dot\psi\sin\theta\sin\phi + \dot\theta\cos\phi)\cos\phi.\end{aligned}\right\} \qquad (e)$$

For the component H_ζ, we have, from expressions (h) of the preceding article,

$$H_\zeta = I_\zeta(\dot\phi + \dot\psi\cos\theta). \qquad (f)$$

Substituting expressions (e) and (f) together with the components ω_m', ω_n', ω_ζ' given by expressions (f) of the preceding article into Eqs. (a), we shall get formulas for calculating the gyroscopic moment in each particular case.

Let us assume now that the gyroscope has a high angular velocity

about its own axis, so that the angular velocity $\dot{\phi}$ is large in comparison with $\dot{\psi}$ and $\dot{\theta}$. Keeping then only terms containing $\dot{\phi}$ as a factor, we obtain

$$\frac{dH_m}{dt} = (I_\xi - I_\eta)\dot{\phi}(\dot{\theta}\cos 2\phi + \dot{\psi}\sin\theta\sin 2\phi),$$
$$\frac{dH_n}{dt} = (I_\xi - I_\eta)\dot{\phi}(\dot{\theta}\sin 2\phi - \dot{\psi}\sin\theta\cos 2\phi)$$
$$\frac{dH_\zeta}{dt} = 0.$$

Substituting into Eqs. (a), we obtain

$$\left.\begin{aligned}\mathfrak{M}_m &= -M_m = -[(I_\xi - I_\eta)\dot{\phi}(\dot{\theta}\cos 2\phi + \dot{\psi}\sin\theta\sin 2\phi) + I_\zeta\dot{\phi}\dot{\theta}],\\ \mathfrak{M}_n &= -M_n = -[(I_\xi - I_\eta)\dot{\phi}(\dot{\theta}\sin 2\phi - \dot{\psi}\sin\theta\cos 2\phi) + I_\zeta\dot{\psi}\dot{\phi}\sin\theta],\\ \mathfrak{M}_\zeta &= -M_\zeta = 0.\end{aligned}\right\} \quad (g)$$

In a particular case of regular precession, $\dot{\theta}$ vanishes and we obtain

$$\left.\begin{aligned}\mathfrak{M}_m &= -M_m = -(I_\xi - I_\eta)\dot{\phi}\dot{\psi}\sin\theta\sin 2\phi,\\ \mathfrak{M}_n &= -M_n = (I_\xi - I_\eta)\dot{\phi}\dot{\psi}\sin\theta\cos 2\phi + I_\zeta\dot{\phi}\dot{\psi}\sin\theta,\\ \mathfrak{M}_\zeta &= -M_\zeta = 0.\end{aligned}\right\} \quad (173)$$

We see that the gyroscopic moment for an unsymmetrical gyroscope consists of two parts: (1) the part $I_\zeta\dot{\phi}\dot{\psi}\sin\theta$, which is of the same kind as we obtained previously for a symmetrical gyroscope, and (2) the part containing $\sin 2\phi$ and $\cos 2\phi$ as factors. Since the gyroscope rotates with high speed about its own axis, this latter part represents a high-frequency pulsating moment acting on the bearings of the axle of the gyroscope. In the case of a two-blade airplane propeller, this pulsating moment may produce dangerous vibrations if its frequency, equal to twice the r.p.m. of the propeller, happens to coincide with the natural frequency of some portion of the airplane structure.

To get some idea of the magnitudes of such gyroscopic moments, let us consider a two-blade propeller of diameter $d = 10$ ft., weight $W = 100$ lb., and rotating at 1,800 r.p.m. Taking the radius of gyration $i = 2.5$ ft. and neglecting the lateral dimensions of the blade, we have, approximately,

$$I_\xi = I_\zeta = \tfrac{100}{386}(30)^2 = 233 \text{ lb.-sec.}^2\text{-in.}, \qquad I_\eta = 0.$$

Then for a horizontal turn of the airplane with angular velocity $\dot{\psi} = 0.5$ rad. per sec., $\theta = \pi/2$, and $\dot{\phi} = 60\pi$ rad. per sec., we have

$$I_\zeta\dot{\phi}\dot{\psi}\sin\theta = 22{,}000 \text{ in.-lb.},$$

and Eqs. (173) give

$$\mathfrak{M}_m = -22{,}000 \sin 2\phi \text{ in.-lb.},$$
$$\mathfrak{M}_n = +22{,}000(1 + \cos 2\phi) \text{ in.-lb.}$$

49. The Gyroscopic Compass.—One of the most important applications of gyroscopes is the gyroscopic compass. Leon Foucault, the originator in the study of the gyroscope, first suggested its use as a compass instead of the magnetic needle.[1] However, many difficulties were encountered in the practical application of this idea, and the first satisfactory gyroscopic compass was accomplished much later by Anschütz and Co.[2] and found application in the German Navy. In this article, we shall discuss the compass developed by the Sperry Gyroscope Co.[3] and used in the United States Navy and in some other countries.

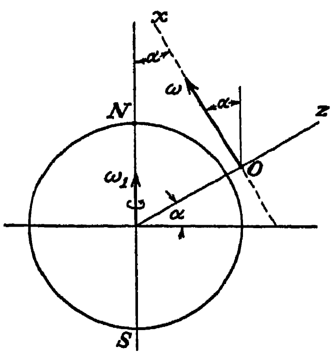

FIG. 267.

The essential part of this compass is a gyroscope with three degrees of freedom, similar to that shown in Fig. 244*b*, page 326. Referring to Fig. 267, we assume that in a position with latitude α, the gyroscope is placed with the axis O_2O_2 of its outer frame directed vertically and its axis of symmetry OO horizontally in the meridian plane. The vertical direction we take for the z-axis and the northward horizontal direction for the x-axis. Then the y-axis will be perpendicular to the plane of the figure and positive in the direction from east to west. This system of axes is no longer immovable, since it participates in the motion of the earth; but if the center of gravity of the gyroscope coincides with the origin of coordinates O, we can still use the principle of angular momentum as discussed on page 332 and shall come to the same equations of motion (161) and (162) as before. If a high angular velocity ω be given to the gyroscope about its axis, it will stably retain the direction of this axis in space. If the axis was directed toward a certain fixed star, it will, in the absence of friction, always continue to be directed toward the same star and will not maintain the direction of the meridian of the place. To indicate the meridian, the axis of the gyroscope must be made to participate in the rotation of the earth and describe in 24 hr. a cone with the angle 2α at the apex (Fig. 267). That is, we must have regular precession of the gyroscope with the velocity

$$\omega_1 = \frac{2\pi}{24 \times 60 \times 60}\,\text{sec}^{-1} = 7.27 \cdot 10^{-5}\,\text{sec}^{-1}.$$

To produce such a precession, a constant moment must act on the gyro-

[1] LEON FOUCAULT, *Compt. rend.*, vol. 35, pp. 421, 424, 469, 1852.

[2] See M. SCHULER, "Der Kreiselkompass von Anschütz und Co.," Kiel, 1910.

[3] Elmer A. Sperry, U.S. Patent 1,279,471.

scope with respect to the axis O_1O_1 of the inner frame in Fig. 244*b*. The magnitude of this moment, equal to the rate of change of the angular momentum of the gyroscope, will be $I\omega\omega_1 \sin \alpha$. If this constant moment is acting, the regular precession of the gyroscope will have the above calculated velocity ω_1 and the axis of the gyroscope always will be in the direction of the meridian of the place; thus the gyroscope can be used as the magnetic needle of a compass. In the Sperry compass, the required moment is produced by an eccentric weight w attached to the inner ring of the gyroscope as shown in Fig. 268. There is a semicircular frame nn rigidly attached to the inner ring, which can freely rotate about the axis O_1O_1 mounted in the outer ring. To this frame, a weight w is attached at the distance a from the axis O_1O_1. If the inner ring rotates about its axis O_1O_1 from its horizontal position by a small angle β_0 (Fig. 269), the weight w furnishes about that axis the moment $wa\beta_0$. The required *angle of elevation* β_0 will now be found from the equation (see Fig. 269)

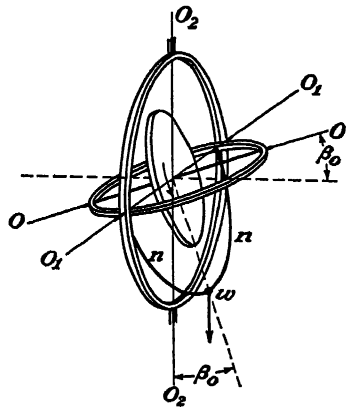

FIG. 268.

$$wa \sin \beta_0 = I\omega\omega_1 \sin(\alpha - \beta_0), \qquad (a)$$

from which

$$\tan \beta_0 = \frac{I\omega\omega_1 \sin \alpha}{wa + I\omega\omega_1 \cos \alpha}. \qquad (b)$$

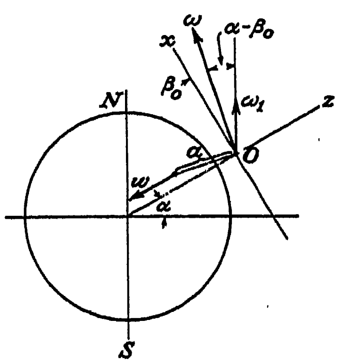

FIG. 269.

In the Sperry compass, the angular velocity $\omega \approx 8{,}500$ r.p.m. and the dimensions of the gyroscope are such that

$$\frac{I\omega\omega_1}{wa} \approx 0.00182.$$

The second term in the denominator of expression (*b*) is small in comparison with the first term, so that with sufficient accuracy we can put

$$\beta_0 \approx 0.00182 \sin \alpha. \qquad (c)$$

We see that only a very small angle of elevation is required to produce the regular precession of the gyroscope which always keeps its axis in the meridional plane of the earth so that it can be used as a compass.

Naturally, this property of the gyroscope may be of practical value only if the direction of the axis of the gyroscope, determined by Eq. (*b*), represents a stable position of that axis; *i.e.*, if an accidental impulse produces a small deviation from this direction, there must be a tendency of the axis to return to its initial direction. In discussing this question of stability, we shall use the method of small vibration already employed in Art. 46. Using again for the moving axes OM, ON, and $O\zeta$ (Fig. 270) and considering the system of axes xyz as fixed, we find as before [see Eqs. (*a*), page 341] that the projections of the angular velocity of the trihedron $OMN\zeta$ on the moving axes are

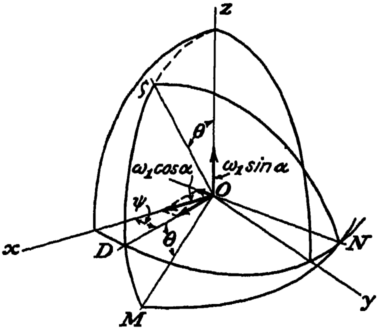

FIG. 270.

$$-\dot{\psi}\sin\theta, \qquad \dot{\theta}, \qquad \dot{\psi}\cos\theta. \qquad (d)$$

Now, however, the x-, y-, z-axes are no longer fixed, and the trihedron $Oxyz$ partcipates in the rotation of the earth with angular velocity ω_1, which gives the components $\omega_1\cos\alpha$ and $\omega_1\sin\alpha$ along the x- and z-axes as can be seen from Fig. 269. Resolving these components first along OD and ON as shown in Fig. 270 and then projecting them on the moving axes OM, ON, and $O\zeta$, we obtain

$$\left.\begin{aligned} \omega_1\cos\alpha\cos\psi\cos\theta - \omega_1\sin\alpha\sin\theta,\\ -\omega_1\cos\alpha\sin\psi,\\ \omega_1\cos\alpha\cos\psi\sin\theta + \omega_1\sin\alpha\cos\theta. \end{aligned}\right\} \qquad (e)$$

Adding these to the components (*d*), we finally obtain the following components of the angular velocity of the trihedron $OMN\zeta$:

$$\left.\begin{aligned} \omega_m' &= -\dot{\psi}\sin\theta + \omega_1\cos\alpha\cos\psi\cos\theta - \omega_1\sin\alpha\sin\theta,\\ \omega_n' &= \dot{\theta} - \omega_1\cos\alpha\sin\psi,\\ \omega_\zeta' &= \dot{\psi}\cos\theta + \omega_1\cos\alpha\cos\psi\sin\theta + \omega_1\sin\alpha\cos\theta. \end{aligned}\right\} \qquad (f)$$

As already explained, the angle θ is very close to $\pi/2$ and the angle ψ is very small.[1] In such a case, by introducing the notation $\beta = (\pi/2) - \theta$ we can use, instead of Eqs. (*f*), the simplified formulas

$$\left.\begin{aligned} \omega_m' &= -\dot{\psi} + \omega_1\beta\cos\alpha - \omega_1\sin\alpha,\\ \omega_n' &= -\dot{\beta} - \omega_1\psi\cos\alpha,\\ \omega_\zeta' &= \omega_1\cos\alpha + \omega_1\beta\sin\alpha. \end{aligned}\right\} \qquad (g)$$

To obtain finally the components of the angular velocity of the gyroscope itself, we have to consider, in addition, the angular velocity ω with respect to its own axis. This gives

$$\left.\begin{aligned} \omega_m &= -\dot{\psi} + \omega_1\beta\cos\alpha - \omega_1\sin\alpha,\\ \omega_n &= -\dot{\beta} - \omega_1\psi\cos\alpha,\\ \omega_\zeta &= \omega + \omega_1\cos\alpha + \omega_1\beta\sin\alpha. \end{aligned}\right\} \qquad (h)$$

The components of the angular momentum of the gyroscope then are

$$\left.\begin{aligned} H_m &= I_1(-\dot{\psi} + \omega_1\beta\cos\alpha - \omega_1\sin\alpha),\\ H_n &= I_1(-\dot{\beta} - \omega_1\psi\cos\alpha),\\ H_\zeta &= I(\omega + \omega_1\cos\alpha + \omega_1\beta\sin\alpha). \end{aligned}\right\} \qquad (i)$$

[1] We assume that any accidental impulse produces only a very small deviation of the gyroscopic axis from the position described in Fig. 269.

Using expressions (g) and (i) in the general equations (162), we obtain the equations of motion for the gyroscope placed as shown in Fig. 269 and subjected to a small impulse at the instant $t = 0$. These equations can be greatly simplified if we observe that ω is very large in comparison with ω_1, $\dot{\beta}$, and $\dot{\psi}$. Then, instead of Eqs. (i), we take

$$H_m = 0, \qquad H_n = 0, \qquad H_\zeta = I\omega, \tag{j}$$

which represent the usual assumption of elementary gyroscopic theory, in which the angular momentum of the gyroscope is represented by a vector directed along its axis of spin. Equations (162) then give

$$\left.\begin{aligned} I\omega(-\dot{\beta} - \omega_1\psi \cos \alpha) &= M_m, \\ I\omega(\dot{\psi} - \omega_1\beta \cos \alpha + \omega_1 \sin \alpha) &= M_n. \end{aligned}\right\} \tag{k}$$

In our case, we need to consider only the moment of the weight w. Hence $M_m = 0$, $M_n = wa\beta$, and Eqs. (k) give

$$\left.\begin{aligned} \dot{\beta} + \omega_1\psi \cos \alpha &= 0, \\ I\omega\dot{\psi} - (I\omega\omega_1 \cos \alpha + wa)\beta &= -I\omega\omega_1 \sin \alpha. \end{aligned}\right\} \tag{l}$$

We already have seen that for the Sperry gyroscope the quantity $I\omega\omega_1$ is small in comparison with wa and can be neglected. Then equations (l) give

$$\left.\begin{aligned} \dot{\beta} + \omega_1\psi \cos \alpha &= 0, \\ \dot{\psi} - \frac{wa}{I\omega}\beta &= -\omega_1 \sin \alpha. \end{aligned}\right\} \tag{m}$$

Eliminating the angle β from these equations, we obtain

$$\ddot{\psi} + \frac{wa\omega_1}{I\omega} \cos \alpha \cdot \psi = 0.$$

From this, we see that the axis of the gyroscope performs simple harmonic oscillations in the horizontal plane, the angular frequency of which is

$$p = \sqrt{\frac{wa\omega_1}{I\omega} \cos \alpha}, \tag{n}$$

and we can write

$$\psi = C_1 \cos pt + C_2 \sin pt. \tag{o}$$

Substituting in the second of Eqs. (m), we find

$$\beta = \frac{I\omega\omega_1 \sin \alpha}{wa} + \frac{I\omega p}{wa}(-C_1 \sin pt + C_2 \cos pt). \tag{p}$$

This shows that the axis of the gyroscope also performs oscillations in the vertical plane about the position indicated in Fig. 269 and defined by the angle

$$\beta_0 = \frac{I\omega\omega_1 \sin \alpha}{wa},$$

which was previously calculated [see Eq. (c)]. From this discussion, we conclude that the Sperry gyroscope stably retains the direction of its axis in the meridian plane and can be used as a compass.

From Eq. (n), we see also that the period of oscillation of the gyroscope is

$$\tau = \frac{2\pi}{p} = 2\pi \sqrt{\frac{I\omega}{wa\omega_1 \cos \alpha}}.$$

With the previously given numerical data, we obtain

$$\tau = 2\pi\sqrt{\frac{0.00182}{\omega_1^2\cos\alpha}}.$$

For $\alpha = 60$ deg., $\tau \approx 5{,}230$ sec. $= 1$ hr. 27 min., which represents very slow oscillations.

In the above calculations, the simplified expressions (j) for the components of the angular momentum were used. A more accurate investigation, by using expressions (i), will not affect our conclusion regarding stability and will show only that on the slow oscillations represented by Eqs. (o) and (p), high-frequency small vibrations will be superimposed.

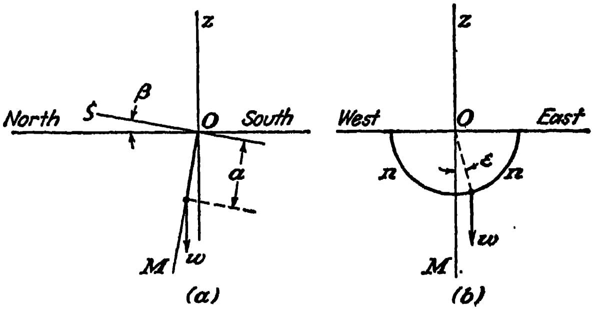

FIG. 271.

The above discussion shows that an accidental impulse produces slow harmonic oscillations of the axis of the gyroscope. To improve the instrument, some damping of these oscillations should be introduced. This damping is accomplished by displacing the weight w along the frame nn in Fig. 268 somewhat to the east from the OM-axis as shown in Fig. 271b. Denoting the small angle corresponding to this displacement by ϵ, we see that the component $w\beta$ of the weight w perpendicular to the axis OM is at the distance $a\epsilon$ from this axis and gives, with respect thereto, a moment

$$M_m = w\beta a\epsilon.$$

Substituting this into Eqs. (k), we obtain, instead of Eqs. (m), the following equations:

$$\dot\beta + \frac{w\beta a\epsilon}{I\omega} + \omega_1\psi\cos\alpha = 0,$$

$$\dot\psi - \frac{wa}{I\omega}\beta = -\omega_1\sin\alpha.$$

Differentiating the second of these equations with respect to time and eliminating β, we obtain

$$\ddot\psi + \frac{wa\epsilon}{I\omega}\dot\psi + \frac{wa\omega_1\cos\alpha}{I\omega}\psi = -\frac{wa\epsilon\omega_1\sin\alpha}{I\omega}.$$

Introducing the notations

$$\frac{wa\epsilon}{I\omega} = 2n, \qquad \frac{wa\omega_1\cos\alpha}{I\omega} = \lambda^2,$$

we give to this equation the form

$$\ddot\psi + 2n\dot\psi + \lambda^2\psi = -\epsilon\lambda^2\tan\alpha, \tag{q}$$

which represents vibrations with damping. Due to the presence of the constant term

on the right-hand side of the equation, we shall find that the direction with respect to which the oscillations proceed is slightly displaced from the direction of the meridian. This constant error is eliminated in the Sperry gyroscope by a certain change in the angle of inclination of the frame nn (Fig. 268) to the ζ-axis of the gyroscope. If we take this angle, for the Northern Hemisphere, equal to $(\pi/2) - \beta_0$, the constant moment for the required precession of the gyroscope will be produced for the horizontal position of the gyroscope's axis and the angle of elevation β_0, indicated in Fig. 269, will be eliminated. At the same time, as it can be shown, the directional error, which was obtained from Eq. (q), will also be eliminated.

It was assumed in our discussion that the gyroscopic compass is attached to the earth. If the compass is attached to a moving body such as a ship, impulses produced on the compass by the motion of the body must be considered. Since the period of oscillation of the gyroscope, as we have seen, is very large, the comparatively short-period oscillations of the ship have only a negligible effect on the direction of the gyroscope's axis.

The constant forward speed of the ship, however, may have a more important effect on the functioning of the gyroscope. Imagine a ship moving with a constant speed v in a northerly direction. In such a case, the $Oxyz$ coordinate system in Fig. 269 will have an additional angular velocity about the y-axis of the magnitude v/R, where R is the radius of the earth. Taking this velocity into account in the same manner as the angular velocity ω_1 of the earth was considered in our previous analysis, we can calculate, for a given latitude and a given v, the deviation of the gyroscopic axis from the meridianal direction. In the Sperry compass, there is a special device to make the proper correction.

In all our discussion, friction forces were neglected. Hence the theoretical conclusions obtained in this manner may be in satisfactory agreement with actual motion only if every precaution is taken to reduce the friction forces to a minimum.

50. The Gyroscopic Pendulum.—A gyroscope suspended at point O above its center of gravity C as shown in Fig. 272 is called a *gyroscopic pendulum*. The distance $\overline{OC}$ we denote by l and take the immovable axes x, y, z, as shown in the figure. Then the position of the moving trihedron $OMN\zeta$ we define, as before, by the angles ψ and θ, the ζ-axis being the axis of the gyroscope and ON the perpendicular to the $zO\zeta$-plane. If, initially, the moving axes coincide with the immovable system of coordinates x,y,z, their position, as shown in the figure, is obtained by rotation first about the z-axis by the angle ψ to the position $ODNz$ and then by rotation about the ON-axis by the angle θ. The angular velocity of the trihedron $OMN\zeta$ is then given by the components $\dot{\psi}$ and $\dot{\theta}$ directed along the z- and ON-axes, respectively. Projecting these components on the moving axes $OMN\zeta$, we obtain

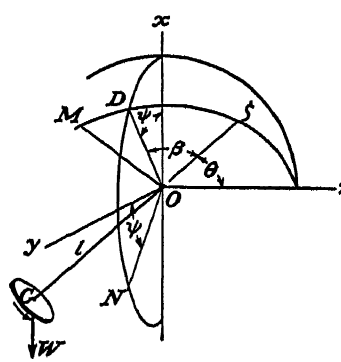

FIG. 272.

$$\omega_m' = -\dot{\psi} \sin \theta, \qquad \omega_n' = \dot{\theta}, \qquad \omega_\zeta' = \dot{\psi} \cos \theta. \tag{a}$$

To obtain the components on the same axes of the angular velocity of the gyroscope, we have to consider, in addition to expressions (a), the angular velocity of the gyroscope with respect to the trihedron $OMN\zeta$. Denoting, as before, by ϕ the angle of rotation of the gyroscope about its ζ-axis, we obtain

$$\omega_m = -\dot{\psi} \sin \theta, \qquad \omega_n = \dot{\theta}, \qquad \omega_\zeta = \dot{\phi} + \dot{\psi} \cos \theta. \tag{b}$$

The corresponding components of the angular momentum of the gyroscope then are

$$H_m = -I_1\dot{\psi} \sin \theta, \qquad H_n = I_1\dot{\theta}, \qquad H_\zeta = I(\dot{\phi} + \dot{\psi} \cos \theta). \tag{c}$$

Considering now only small oscillations of the pendulum with respect to its vertical position, the angle ψ will be a small angle and θ will be different from $\pi/2$ only by a small quantity. Introducing the notation

$$\beta = \frac{\pi}{2} - \theta$$

and considering ψ, β, $\dot{\psi}$, $\dot{\beta}$ all as small quantities, we obtain, after neglecting small quantities to the powers higher than the first,

$$\left.\begin{aligned} \omega_m' &= -\dot{\psi}, & \omega_n' &= -\dot{\beta}, & \omega_\zeta' &= 0. \\ H_m &= -I_1\dot{\psi}, & H_n &= -I_1\dot{\beta}, & H_\zeta &= I\dot{\phi}. \end{aligned}\right\} \tag{d}$$

Substituting these values into Eqs. (162), we obtain

$$\left.\begin{aligned} -I_1\ddot{\psi} - I\dot{\phi}\dot{\beta} &= M_m, \\ -I_1\ddot{\beta} + I\dot{\phi}\dot{\psi} &= M_n, \\ I\ddot{\phi} &= M_\zeta. \end{aligned}\right\} \tag{e}$$

Let us assume now that the moment M_ζ vanishes.[1] Then the angular velocity $\dot{\phi}$ of the gyroscope about its own axis remains constant; and denoting this by ω, we obtain from the first two of Eqs. (e)

$$\left.\begin{aligned} I_1\ddot{\psi} + I\omega\dot{\beta} &= -M_m, \\ I_1\ddot{\beta} - I\omega\dot{\psi} &= -M_n. \end{aligned}\right\} \tag{f}$$

The moments on the right-hand sides of these equations are produced in this case by the gravity force W of the gyroscope as can be seen from Fig. 272. In deriving expressions for these moments, we first calculate the moments of the gravity force with respect to the fixed axes x,y,z. Observing that for small values of the angles ψ and β the coordinates of the center of gravity C are approximately

$$x_c = -l, \qquad y_c = -l\psi, \qquad z_c = -l\beta,$$

[1] This means that we neglect friction in the bearings of the gyroscope or that a torque, compensating that friction, is always supplied.

we obtain

$$M_x = 0, \qquad M_y = Wl\beta, \qquad M_z = -Wl\psi.$$

Projecting these components of the external moment on the axes OM and ON, we find, for small values of the angles ψ and β,

$$M_m = Wl\psi, \qquad M_n = Wl\beta,$$

and Eqs. (f) give

$$\left.\begin{aligned} I_1\ddot{\psi} + I\omega\dot{\beta} &= -Wl\psi, \\ I_1\ddot{\beta} - I\omega\dot{\psi} &= -Wl\beta. \end{aligned}\right\} \tag{g}$$

These equations define the motion for small oscillations of the gyroscopic pendulum. If we assume that the angular velocity ω of the gyroscope vanishes, we obtain the following known equations for a spherical pendulum without any gyroscopic action;

$$\left.\begin{aligned} I_1\ddot{\psi} + Wl\psi &= 0, \\ I_1\ddot{\beta} + Wl\beta &= 0. \end{aligned}\right\} \tag{h}$$

Returning now to Eqs. (g), we solve the first equation for $\dot{\beta}$ and by differentiation obtain also $\ddot{\beta}$. Then differentiating the second equation and substituting the expressions for $\dot{\beta}$ and $\ddot{\beta}$, we find

$$I_1^2\psi^{IV} + (I^2\omega^2 + 2I_1Wl)\ddot{\psi} + W^2l^2\psi = 0. \tag{i}$$

Substituting $\psi = e^{mt}$, we obtain the characteristic equation

$$I_1^2m^4 + (I^2\omega^2 + 2I_1Wl)m^2 + W^2l^2 = 0, \tag{k}$$

from which

$$m^2 = \frac{-(I^2\omega^2 + 2I_1Wl) \pm I\omega\sqrt{I^2\omega^2 + 4I_1Wl}}{2I_1^2}.$$

It is seen that the expression under the radical is always positive and smaller than $(I^2\omega^2 + 2I_1Wl)^2$, which indicates that both values of m^2 are negative. Using for the absolute values of m^2 the notations

$$\left.\begin{aligned} \lambda_1^2 &= \frac{I^2\omega^2 + 2I_1Wl - I\omega\sqrt{I^2\omega^2 + 4I_1Wl}}{2I_1^2}, \\ \lambda_2^2 &= \frac{I^2\omega^2 + 2I_1Wl + I\omega\sqrt{I^2\omega^2 + 4I_1Wl}}{2I_1^2}, \end{aligned}\right\} \tag{l}$$

we find

$$\lambda_1^2\lambda_2^2 = \frac{W^2l^2}{I_1^2},$$

$$\lambda_1^2 + \lambda_2^2 = \frac{I^2\omega^2 + 2I_1Wl}{I_1^2},$$

$$\lambda_1^2 - \lambda_2^2 = -\frac{I\omega\sqrt{I^2\omega^2 + 4I_1Wl}}{I_1^2}.$$

Taking λ_1 and λ_2 both positive and observing that $\lambda_1 < \lambda_2$, we obtain from these equations

$$\lambda_1\lambda_2 = \frac{Wl}{I_1}, \qquad \lambda_1 - \lambda_2 = -\frac{I\omega}{I_1}, \qquad \lambda_1 + \lambda_2 = \frac{\sqrt{I^2\omega^2 + 4I_1Wl}}{I_1},$$

which gives

$$\lambda_1 = \frac{-I\omega + \sqrt{I^2\omega^2 + 4I_1Wl}}{2I_1}, \qquad \lambda_2 = \frac{I\omega + \sqrt{I^2\omega^2 + 4I_1Wl}}{2I_1}. \qquad (m)$$

With these values of λ_1 and λ_2, we find the following four roots of the characteristic equation (k):

$$m_1 = \lambda_1 i, \qquad m_2 = -\lambda_1 i, \qquad m_3 = \lambda_2 i, \qquad m_4 = -\lambda_2 i.$$

The general solution of Eq. (i) then is

$$\psi = C_1\cos\lambda_1 t + C_2\sin\lambda_1 t + C_3\cos\lambda_2 t + C_4\sin\lambda_2 t. \qquad (n)$$

Substituting this into the second of Eqs. (g) and eliminating $\ddot{\beta}$ by the use of the first equation, we obtain

$$\beta = -C_1\sin\lambda_1 t + C_2\cos\lambda_1 t + C_3\sin\lambda_2 t - C_4\cos\lambda_2 t. \qquad (o)$$

We see that there are two different frequencies of oscillation as given by Eqs. (m). If the angular velocity ω is very large, we can consider the second term under the radical in Eqs. (m) as small in comparison with the first and write

$$\lambda_1 \approx \frac{Wl}{I\omega}, \qquad \lambda_2 \approx \frac{I\omega}{I_1}. \qquad (p)$$

The first of these two frequencies is very small, while the second is very large.

The constants of integration in the solutions (n) and (o) can be calculated in each particular case if the initial conditions are given. Assume, for example, that for $t = 0$:

$$\psi = 0, \qquad \beta = 0, \qquad \dot{\psi} = \dot{\psi}_0, \qquad \dot{\beta} = 0.$$

This requires that

$$C_1 = -C_3 = 0, \qquad C_2 = C_4 = \frac{\dot{\psi}_0}{\lambda_1 + \lambda_2}.$$

Using expressions (p) for the two frequencies and neglecting λ_1 in comparison with λ_2, we obtain

$$\left.\begin{aligned} \psi &= \frac{\dot{\psi}_0}{\lambda_2}(\sin\lambda_1 t + \sin\lambda_2 t), \\ \beta &= \frac{\dot{\psi}_0}{\lambda_2}(\cos\lambda_1 t - \cos\lambda_2 t). \end{aligned}\right\} \qquad (q)$$

Multiplying these angles by $-l$, we obtain for the coordinates of the center of gravity of the gyroscope,

$$\left.\begin{aligned} y_c &= -\frac{\dot\psi_0 l}{\lambda_2}(\sin \lambda_1 t + \sin \lambda_2 t), \\ z_c &= -\frac{\dot\psi_0 l}{\lambda_2}(\cos \lambda_1 t - \cos \lambda_2 t). \end{aligned}\right\} \qquad (r)$$

The first terms of these expressions represent a slow motion of the center C of the gyroscope around the vertical x-axis along a circle of radius $\dot\psi_0 l/\lambda_2$ with angular velocity λ_1 (Fig. 273a). On this rotation, the

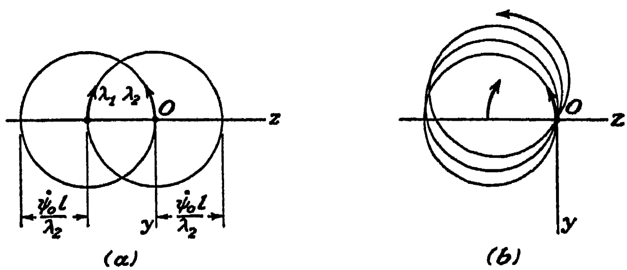

FIG. 273.

motion represented by the second terms in expressions (r) is superimposed. This second motion represents a rotation with high angular velocity λ_2 along a circle of the same radius $\dot\psi_0 l/\lambda_2$ (Fig. 273a). As a result of this superposition, we obtain the trajectory described by the center of gravity C as shown in Fig. 273b.

For comparison, let us consider the spherical pendulum without gyroscopic action ($\omega = 0$). From the equations of motion (h), we conclude that the frequency in this case is

$$\lambda_0 = \sqrt{\frac{Wl}{I_1}};$$

and assuming the same initial conditions as before, we obtain

$$\psi = \frac{\dot\psi_0}{\lambda_0}\sin \lambda_0 t, \qquad \beta = 0. \qquad (s)$$

Observing that in the case of large values of ω, the frequency λ_2 is large in comparison with λ_0 and comparing expressions (q) and (s), we conclude that for the same initial values of $\dot\psi_0$, the oscillations of the gyroscopic pendulum have an amplitude many times smaller than the amplitude of the spherical pendulum. The gyroscopic pendulum retains its vertical position with a greater stability than the spherical pendulum, and its stability increases with the increase of the angular velocity ω of

the gyroscope. This characteristic of the gyroscopic pendulum is used in many applications where it is important to retain stably a vertical line in space as, for example, in the case of instruments used on ships and airplanes to indicate the true horizontal.[1]

In the preceding discussion, the point of suspension O of the gyroscopic pendulum (Fig. 272), was considered as fixed. Assume now that this point performs harmonic oscillations. Such a condition we have, for example, in using the gyroscopic pendulum on a rolling ship where, owing to oscillation of the ship, forced vibrations of the pendulum are produced. In studying these vibrations, we assume that the point O is moving along the z-axis and that this motion is given by the equation

$$z = a \sin pt. \tag{t}$$

Then the coordinate system $Oxyz$ in Fig. 272 is no longer immovable but performs the oscillations (t). To take care of this second motion and investigate only the relative motion of the pendulum with respect to the x-, y-, z-axes, we add to the particles of our system the inertia forces corresponding to the motion (t) and, after this, treat x,y,z as immovable axes. These inertia forces can be replaced by their resultant applied at the center of gravity C of the gyroscope and equal to

$$R = -\frac{W}{g}\ddot{z} = \frac{W}{g}ap^2 \sin pt.$$

This force, parallel to the z-axis, gives the moments[2]

$$M_x = Ry_c = -Rl\psi, \qquad M_y = Rl, \qquad M_z = 0.$$

Projecting these moments onto the axes $OMN\zeta$ in Fig. 272, we obtain[3]

$$M_m = 0, \qquad M_n = Rl, \qquad M_\zeta = 0.$$

Adding these moments of the inertia force to the previously calculated moments produced by the gravity force, Eqs. (f) give

$$\left.\begin{aligned} I_1\ddot{\psi} + I\omega\dot{\beta} &= -Wl\psi, \\ I_1\ddot{\beta} - I\omega\dot{\psi} &= -Wl\beta - Rl = -Wl\beta - \frac{W}{g}ap^2 l \sin pt. \end{aligned}\right\} \tag{u}$$

[1] The description of some of these instruments can be found in the article by O. Martienssen mentioned before, see p. 326.

[2] It is assumed, as before, that oscillations of the pendulum are small.

[3] In this calculation, we first project M_x, M_y, M_z on the axes OD, ON, and Oz and obtain $M_d = M_x \cos\psi + M_y \sin\psi \approx 0$, $M_n = M_y \cos\psi - M_x \sin\psi \approx Rl$, $M_z = 0$. Then projecting M_d onto the axes OM and $O\zeta$, we find $M_m = M_d \cos\beta \approx 0$, $M_\zeta \approx M_d \sin\beta \approx 0$.

The particular solution of these equations is

$$\psi = A \cos pt, \qquad \beta = B \sin pt. \tag{v}$$

To calculate the amplitudes A and B, we substitute expressions (v) into Eqs. (u), which gives

$$\begin{aligned} (Wl - I_1p^2)A + I\omega pB &= 0, \\ I\omega pA + (Wl - I_1p^2)B &= -\frac{W}{g}ap^2l. \end{aligned}$$

From these equations, we find the following amplitudes of forced vibration of the gyroscopic pendulum:

$$\left.\begin{aligned} A &= -\frac{Walp^2I\omega p}{g[I^2\omega^2p^2 - (Wl - I_1p^2)^2]}, \\ B &= \frac{Walp^2(Wl - I_1p^2)}{g[I^2\omega^2p^2 - (Wl - I_1p^2)^2]}. \end{aligned}\right\} \tag{w}$$

The condition of resonance is obtained when the denominator in these expressions vanishes, which gives

$$I^2\omega^2p^2 - (Wl - I_1p^2)^2 = 0$$

or

$$I_1{}^2p^4 - (I^2\omega^2 + 2WlI_1)p^2 + W^2l^2 = 0.$$

Comparing this equation with the characteristic equation (k) and using notations (l), we conclude that the two critical frequencies are

$$p_1 = \lambda_1 \qquad \text{and} \qquad p_2 = \lambda_2,$$

where λ_1 and λ_2 are given by expressions (m). In the case of high-speed gyroscopes, we can use the simplified expressions (p) instead of expressions (m) and conclude that one of the two critical frequencies is very small while the other is very large.

If we substitute $\omega = 0$ in Eqs. (u), we obtain the case of a spherical pendulum and Eqs. (w) give

$$A = 0, \qquad B = -\frac{Walp^2}{g(Wl - I_1p^2)}.$$

Forced vibrations in this case are produced in the same plane in which the disturbing force is acting, and the critical frequency p_0 is obtained from the equation

$$Wl - I_1p_0{}^2 = 0,$$

from which

$$p_0 = \sqrt{\frac{Wl}{I_1}}.$$

In the case of a gyroscopic pendulum, both amplitudes A and B,

as given by Eqs. (w), are usually different from zero and forced oscillations of the gyroscope occur in both directions. In a particular case where $p^2 = Wl/I_1$, we have $B = 0$ and forced vibrations are produced in the plane perpendicular to that in which the disturbing forces are acting.

Usually the angular velocity ω of the gyroscope is very large in comparison with the frequency p of the disturbing force and with the frequency p_0 of the spherical pendulum. In such case, we can neglect, in the denominators of expressions (w), all terms except those containing ω^2 as a factor and obtain

$$A = -\frac{Walp}{gI\omega} = -\frac{ap_0^2p}{g\lambda_2},$$
$$B = \frac{Wal(Wl - I_1p^2)}{gI^2\omega^2} = \frac{ap_0^2(p_0^2 - p^2)}{g\lambda_2^2}.$$

Since λ_2 in this case is a large number [see Eq. (p)], we conclude that the amplitudes of forced vibration are small and the gyroscopic pendulum, under the action of a periodic disturbing force, stably retains its vertical position.

51. The Gyroscopic Ship Stabilizer.—The idea of using the gyroscope for stabilization of ships was first suggested by Otto Schlick.[1] It was proposed to use for such stabilization a heavy gyroscope G (Fig. 274) the vertical axis OO of which is mounted in the frame NN free to rotate about the axis O_1O_1. The center of gravity of the frame, together with the gyroscope, is below the axis O_1O_1, so that under statical conditions, the frame stably retains its position in a vertical plane perpendicular to the longitudinal axis of the ship. From an elementary consideration of changes in the angular momentum of the gyroscope, it can be concluded that any rolling of the ship will result in oscillations of the frame NN about the axis O_1O_1. These oscillations are reduced by applying brakes against the wheel MM. The energy brought to the rolling ship by the periodic action of waves is thus dissipated in the form of heat generated by the friction forces. Experience shows that in this manner, a great reduction in rolling oscillations of the ship can be accomplished.

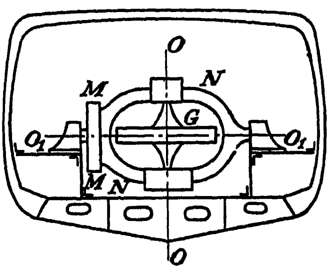

FIG. 274.

[1] Institution of Naval Architects, March, 1904. The theory of this stabilization was developed by A. Föppl, *Z. Ver. deut. Ing.*, vol. 48, p. 478, 1904. See also his "Technische Mechanik," vol. 6, 1910, and the "Theorie des kreisels," by F. Klein and A. Sommerfeld, vol. 4, p. 794, 1910.

In the derivation of equations of motion of the *stabilizer*, it is necessary to consider not only the mass of the gyroscope itself, as we did before, but also the mass of the frame NN and of the ship. The friction forces, usually neglected in our previous problems, are also of importance in this case and must be considered. The derivation of the required equations can be greatly simplified if, from the start, we assume (1) that the angular velocity ω of the gyroscope is very large, so that its angular momentum can be assumed always directed along the axis of spin, (2) that the mass of the stabilizer is negligible in comparison with the mass of the ship, and (3) that the ship performs only rolling motion. Referring to Fig. 275, the position of the frame NN of the stabilizer can be defined by the angle ψ, representing the roll of the ship, and the angle $\beta = (\pi/2) - \theta$, representing rotation of the frame NN with respect to its axis O_1O_1. In our further discussion, both these angles will be considered as small.

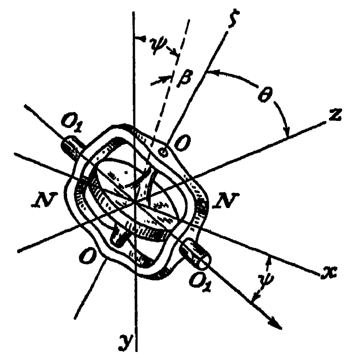

FIG. 275.

Considering first the free rolling oscillations of the ship, without the action of the stabilizer, we can use the known equation for damped oscillations (see page 33)

$$J\ddot{\psi} + C\dot{\psi} + WL\psi = 0, \qquad (a)$$

in which J is the moment of inertia of the ship with respect to its longitudinal axis Oz, C is the damping constant giving the magnitude of the moment of resisting forces about the z-axis when $\dot{\psi}$ is equal to unity, W is the weight of the ship, and L is its *metacentric height.* In a like manner, the free oscillations of the frame NN, when the ship is stationary, are given by the equation

$$j\ddot{\beta} + c\dot{\beta} + wl\beta = 0, \qquad (b)$$

in which j denotes the moment of inertia of the frame NN together with the gyroscope about the axis O_1O_1, c is the magnitude of the moment of friction forces when $\dot{\beta}$ is equal to unity, w is the weight of the frame together with the gyroscope, and l is the distance of their common center of gravity from the axis of rotation O_1O_1. This equation holds independently of rotation of the gyroscope about its own axis as long as the axis O_1O_1 is stationary. The rotation of the gyroscope in this case affects only the pressures on the bearings supporting the axle O_1O_1. The corresponding gyroscopic moment with respect to the axis Oz in

Fig. 275 and transmitted to the ship is

$$\mathfrak{M}_z = -H\dot{\beta}, \tag{c}$$

where $H = I\omega$ denotes the angular momentum of the gyroscope.

Considering now the rolling movement of the ship, while the frame NN does not oscillate, we see that the rate of change of the angular momentum of the gyroscope is $H\dot{\psi}$ and the corresponding gyroscopic moment with respect to the axis O_1O_1 in Fig. 275 is

$$\mathfrak{M}_n = -H\dot{\psi}. \tag{d}$$

This moment represents the action of the axle OO of the gyroscope on its bearings in the frame NN.

After this preliminary discussion, we can write equations of motion for the case when the ship and the frame of the stabilizer are oscillating simultaneously. In writing the equation of motion for the rolling ship, we have to take into consideration the gyroscopic moment (c), representing the action of the stabilizer on the ship; while in the equation for oscillation of the frame, we have to consider the gyroscopic moment (d). In this manner we obtain

$$\left.\begin{aligned} J\ddot{\psi} + C\dot{\psi} + WL\psi &= -H\dot{\beta}, \\ j\ddot{\beta} + c\dot{\beta} + wl\beta &= H\dot{\psi}. \end{aligned}\right\} \tag{e}$$

The sign of the right-hand side of the second equation is opposite to the sign of the gyroscopic moment (d), since the angle β in Fig. 275 is measured in the direction opposite to positive rotation about the axis O_1O_1.

Equations (e) define the simultaneous oscillations of the ship and of the stabilizer. In their derivation, we considered only rolling of the ship, but it may be seen that the reactions of the brakes give a couple acting on the ship in the plane of the wheel MM in Fig. 274 and will produce pitching of the ship. Due to the fact that the moment of inertia of the ship with respect to its transverse axis is very large in comparison with that about the longitudinal axis, the pitching oscillations are very small and can be neglected.[1] A more important effect on the motion of the system may result from rotation of the ship about its vertical axis, since such rotation may affect rolling of the ship.[2]

We begin our study of the simultaneous motions of the ship and stabilizer with the case where friction forces can be neglected. Equations (e) then give

[1] A. Föppl, *Z. Ver. deut. Ing.*, vol. 50, p. 983, 1906.

[2] This problem was discussed by M. Schuler, *Z. Ver. deut. Ing.*, vol. 68, p. 1224, 1924.

$$\left.\begin{aligned} J\ddot{\psi} + WL\psi + H\dot{\beta} &= 0, \\ j\ddot{\beta} + wl\beta - H\dot{\psi} &= 0. \end{aligned}\right\} \tag{f}$$

Eliminating β from these equations, we obtain

$$Jj\psi^{\mathrm{IV}} + (WLj + wlJ + H^2)\ddot{\psi} + wlWL\psi = 0. \tag{g}$$

The corresponding characteristic equation is

$$Jjm^4 + (WLj + wlJ + H^2)m^2 + wlWL = 0. \tag{h}$$

Introducing the notation

$$WLj + wlJ + H^2 = k, \tag{i}$$

we find

$$m^2 = \frac{-k \pm \sqrt{k^2 - 4JjwlWL}}{2Jj} \tag{j}$$

From notation (i) it can be concluded that

$$k^2 - 4JjwlWL > (WLj + wlJ)^2 - 4JjwlWL = (WLj - wlJ)^2.$$

Hence, the expression under the radical in Eq. (j) is always positive and m^2 is negative. All four roots of the characteristic equation (h) are imaginary numbers; and with the notations

$$\left.\begin{aligned} p_1^2 &= \frac{k - \sqrt{k^2 - 4JjwlWL}}{2Jj}, \\ p_2^2 &= \frac{k + \sqrt{k^2 - 4JjwlWL}}{2Jj}, \end{aligned}\right\} \tag{k}$$

we can represent the solution of Eq. (g) in the following form:

$$\psi = C_1\cos p_1t + C_2\sin p_1t + C_3\cos p_2t + C_4\sin p_2t. \tag{l}$$

To obtain the corresponding expression for β, we calculate, from the first of Eqs. (f),

$$\begin{aligned} H\dot{\beta} &= -(J\ddot{\psi} + WL\psi), \\ H\ddot{\beta} &= -(J\dddot{\psi} + WL\dot{\psi}). \end{aligned}$$

Substituting the expression for $\ddot{\beta}$ into the second of Eqs. (f), we obtain

$$Hwl\beta = Jj\dddot{\psi} + (WLj + H^2)\dot{\psi}.$$

Using now for ψ its expression (l), we obtain

$$\beta = D_1\cos p_1t + D_2\sin p_1t + D_3\cos p_2t + D_4\sin p_2t, \tag{m}$$

where

$$D_1 = C_2p_1\frac{WLj + H^2 - Jjp_1^2}{Hwl}, \qquad D_2 = -C_1p_1\frac{WLj + H^2 - Jjp_1^2}{Hwl},$$

$$D_3 = C_4p_2\frac{WLj + H^2 - Jjp_2^2}{Hwl}, \qquad D_4 = -C_3p_2\frac{WLj + H^2 - Jjp_2^2}{Hwl}.$$

Expressions (l) and (m) represent the complete solution of Eqs. (f). We obtained two harmonic oscillations of the frequencies p_1 and p_2 as given by expressions (k). The four constants of integration $C_1 \ldots C_4$ can be calculated in each particular case if the initial values of the angles ψ_0 and β_0 and the initial velocities $\dot{\psi}_0$ and $\dot{\beta}_0$ are given.

When ω, the angular velocity of the gyroscope, is very large, the quantity $H^2 = I^2\omega^2$ is large in comparison with WLj and wlJ and $k \approx H^2$ is large in comparison with $4JjwlWL$. The formulas (k) then give

$$p_1 \approx \frac{\sqrt{wlWL}}{H}, \qquad p_2 \approx \frac{H}{\sqrt{Jj}}. \tag{n}$$

The first of these frequencies is very small, while the second is very large. We conclude that in the case of free oscillations of the system, high-frequency vibrations are superposed on the slow fundamental oscillations.

If, by some constraint, the frame NN of the stabilizer in Fig. 274 is fixed, there will be only rolling of the ship and the corresponding equation of motion is obtained from the first of Eqs. (f) by putting $\dot{\beta} = 0$, which gives

$$J\ddot{\psi} + WL\psi = 0. \tag{o}$$

In this case, the ship performs harmonic oscillations of the frequency

$$p_0 = \sqrt{\frac{WL}{J}}. \tag{p}$$

It is seen that by introducing the stabilizer, we split this frequency into two parts as given by Eqs. (n): one very small and the other very large. Equations (m), for large values of ω, give

$$\left.\begin{aligned} D_1 &\approx C_2\frac{\lambda_1 H}{wl} \approx C_2\sqrt{\frac{WL}{wl}}, & D_2 &\approx -C_1\sqrt{\frac{WL}{wl}}, \\ D_3 &\approx C_4\lambda_2\frac{WLj + H^2 - k}{Hwl} \approx -C_4\sqrt{\frac{J}{j}}, & D_4 &\approx C_3\sqrt{\frac{J}{j}}, \end{aligned}\right\} \tag{q}$$

and we obtain

$$\left.\begin{aligned} \psi &= C_1\cos p_1t + C_2\sin p_1t + C_3\cos p_2t + C_4\sin p_2t, \\ \beta &= \sqrt{\frac{WL}{wl}}\,(C_2\cos p_1t - C_1\sin p_1t) + \sqrt{\frac{J}{j}}\,(-C_4\cos p_2t + C_3\sin p_2t). \end{aligned}\right\} \tag{r}$$

We see that for the fundamental mode of vibration with the frequency p_1, the ratio of the amplitudes of the angles β and ψ is $\sqrt{WL/wl}$ while for the higher mode of vibration the ratio is $\sqrt{J/j}$.

Let us calculate now the amplitudes of vibration for the particular case in which, by some impulse on a ship at rest, an initial velocity $\dot{\psi}_0$ of

rolling of the ship is produced. The initial conditions for $t = 0$ are

$$\psi = 0, \qquad \dot{\psi} = \dot{\psi}_0, \qquad \beta = 0, \qquad \dot{\beta} = 0.$$

From expressions (r), we then obtain

$$C_1 + C_3 = 0, \qquad C_2 p_1 + C_4 p_2 = \dot{\psi}_0,$$
$$\sqrt{\frac{WL}{wl}}\, C_2 - \sqrt{\frac{J}{j}}\, C_4 = 0, \qquad -\sqrt{\frac{WL}{wl}}\, p_1 C_1 + \sqrt{\frac{J}{j}}\, p_2 C_3 = 0,$$

and the constants of integration are

$$C_1 = 0,$$
$$C_3 = 0,$$
$$C_2 = \frac{\dot{\psi}_0 \sqrt{wlJ}}{p_1\sqrt{wlJ} + p_2\sqrt{WLj}} \approx \frac{\dot{\psi}_0 \sqrt{wlJ}}{p_2\sqrt{WLj}} = \frac{\dot{\psi}_0 J}{H}\sqrt{\frac{wl}{WL}},$$
$$C_4 = \frac{\dot{\psi}_0 \sqrt{WLj}}{p_1\sqrt{wlJ} + p_2\sqrt{WLj}} \approx \frac{\dot{\psi}_0}{H}\sqrt{Jj} = \frac{\dot{\psi}_0 J}{H}\sqrt{\frac{j}{J}}.$$

Substituting these values into expressions (r), we obtain

$$\psi = \frac{\dot{\psi}_0 J}{H}\left(\sqrt{\frac{wl}{WL}}\sin p_1 t + \sqrt{\frac{j}{J}}\sin p_2 t\right),$$
$$\beta = \frac{\dot{\psi}_0 J}{H}(\cos p_1 t - \cos p_2 t).$$

If the gyroscope is at rest ($\omega = 0$) and with the same initial conditions, we obtain, from Eq. (o),

$$\psi = \frac{\dot{\psi}_0}{p_0}\sin p_0 t = \dot{\psi}_0\sqrt{\frac{J}{WL}}\sin p_0 t.$$

We see that owing to the action of the gyroscope, the amplitude of the fundamental mode of vibration is reduced in the ratio

$$\frac{\dot{\psi}_0 J}{H}\sqrt{\frac{wl}{WL}} : \dot{\psi}_0\sqrt{\frac{J}{WL}} = \frac{\sqrt{Jwl}}{H}.$$

We conclude that with large values of H, a considerable reduction in the amplitude of rolling can be accomplished by introducing the stabilizer.

By using the stabilizer, not only the amplitude of rolling of the ship can be reduced but this oscillation can be damped out. For this purpose, we have only to apply proper friction forces to the wheel MM of the stabilizer (Fig. 274). Assuming viscous friction and neglecting friction between the ship and water, Eqs. (c) give

$$\left.\begin{aligned} J\ddot{\psi} + WL\psi + H\dot{\beta} &= 0, \\ j\ddot{\beta} + c\dot{\beta} + wl\beta - H\dot{\psi} &= 0. \end{aligned}\right\} \qquad (s)$$

Eliminating β as before, we obtain for ψ a differential equation of the fourth order, the solution of which could be represented in the form[1]

$$\psi = e^{-n_1 t}(C_1 \cos p_1 t + C_2 \sin p_1 t) + e^{-n_2 t}(C_3 \cos p_2 t + C_4 \sin p_2 t).$$

We see again oscillations of two different frequencies; but owing to damping, these oscillations gradually die out. Considering the practically more important fundamental type of oscillation with the frequency p_1, we see that the damping depends on the constant n_1, which can be calculated in each particular case if the constant c in the second of Eqs. (s) is known. If this latter constant vanishes, we have no damping. Again we shall have no damping if the constant c is so large that the frame of the stabilizer does not rotate at all about its axis. In this case, $\beta = 0$ and rolling of the ship is defined by Eq. (o). To find c corresponding to the most efficient damping, the quantity n_1 must be calculated for several values of c and the value giving the largest n_1 should be selected. For illustration, we take as a numerical example[2] the case of a ship having a weight $W = 6{,}000(10)^3$ kg., a metacentric height $L = 0.45$ m., and a rolling period $\tau_0 = 2\pi/p_0 = 15$ sec. For the stabilizer, we assume

$$\begin{aligned} wl &= 10^4 \text{ kg.-m.}, \\ j &= 3{,}000 \text{ kg.-m.-sec.}^2, \\ H &= I\omega = 408(10)^3 \text{ kg.-m.-sec.} \end{aligned}$$

The values of n_1 calculated for several assumed values of c are given in Table XV.

TABLE XV

c, kg.-m.-sec.	8,300	18,500	39,000	47,100	61,200	141,000
n_1, sec^{-1}	0.019	0.041	0.079	0.080	0.074	0.037

We see that n_1 first increases with c and then begins to decrease. Taking $c = 39{,}000$ kg.-m.-sec., for which $n_1 = 0.079$ sec.$^{-1}$, it was found that $p_1 = 0.330\,(\text{sec})^{-1}$, $\tau_1 = 2\pi/p_1 = 19$ sec., $e^{-n_1\tau_1} = e^{-1.50} = 0.22$. This indicates that with the assumed friction, the amplitude of oscillation during one cycle is reduced to 0.22 of its initial magnitude.

Forced rolling of the ship, produced by a periodic wave action, can also be investigated without difficulty. Assume first that there is no damping. Then, instead of Eqs. (f), we obtain in this case,

$$\left.\begin{aligned} J\ddot{\psi} + WL\psi + H\dot{\beta} &= M \sin pt, \\ j\ddot{\beta} + wl\beta - H\dot{\psi} &= 0, \end{aligned}\right\} \qquad (t)$$

where M sin pt represents the moment about the longitudinal axis of

[1] We assume that friction is not large enough to make the motion *aperiodic*.

[2] This example was calculated by A. Föppl (see footnote, p. 366).

the ship produced by the wave action. Considering only forced vibrations, we take the particular solution of Eqs. (t) in the form

$$\psi = \psi_0 \sin pt, \qquad \beta = \beta_0 \cos pt.$$

Substituting these solutions back into Eqs. (t), we obtain

$$\left.\begin{aligned} (WL - Jp^2)\psi_0 - Hp\beta_0 &= M, \\ (wl - jp^2)\beta_0 - Hp\psi_0 &= 0. \end{aligned}\right\} \tag{u}$$

Solving for ψ_0 and β_0, we obtain the amplitudes of forced vibrations as follows:

$$\psi_0 = \frac{M(wl - jp^2)}{N}, \qquad \beta_0 = \frac{MHp}{N}, \tag{v}$$

where

$$\begin{aligned} N &= (WL - Jp^2)(wl - jp^2) - H^2p^2 \\ &= Jjp^4 - (WLj + wlJ + H^2)p^2 + WLwl. \end{aligned}$$

Comparing this expression for N with the left-hand side of Eq. (h), we conclude that N vanishes and the amplitudes (v) become infinitely large if

$$p^2 = p_1^2 \qquad \text{or} \qquad p^2 = p_2^2,$$

where p_1 and p_2 are the two natural frequencies of the system calculated before [see expressions (k)]. This indicates that we obtain the known phenomenon of resonance if the frequency p of the disturbing couple M coincides with one of the natural frequencies of the system.

From the first of expressions (v), we can see also that the amplitude ψ_0 vanishes if

$$wl = jp^2 \qquad \text{or} \qquad p = \sqrt{\frac{wl}{j}}. \tag{w}$$

This specific frequency p is the same as the frequency of oscillation of the frame NN of the stabilizer if considered as a pendulum. It can be obtained from Eq. (b) by making c vanish, and the stabilizer acts in this case as a damper. The amplitude of vibration of the frame is found from the first of Eqs. (u), which gives

$$\beta_0 = -\frac{M}{Hp},$$

and we have

$$\beta = -\frac{M}{Hp}\cos pt.$$

The gyroscopic moment corresponding to this oscillation of the frame is

$$\mathfrak{M}_z = -H\dot{\beta} = -M \sin pt.$$

We see that at each instant, it balances the external disturbing couple $M \sin pt$.

The effect of friction, applied to the wheel MM in Fig. 274, can also be considered in studying the forced vibrations. Instead of Eqs. (t), we obtain in this case,

$$J\ddot{\psi} + WL\psi + H\dot{\beta} = M \sin pt,$$
$$j\ddot{\beta} + c\dot{\beta} + wl\beta - H\dot{\psi} = 0.$$

The particular solution of these equations, representing forced vibrations, we take in the form

$$\psi = A_1\cos pt + A_2\sin pt, \qquad \beta = A_3\cos pt + A_4\sin pt.$$

Substituting into the equations of motion and equating to zero the factors before $\sin pt$ and $\cos pt$, we obtain four equations from which the constants $A_1 \ldots A_4$, defining the amplitudes of forced vibrations, can be calculated in each particular case. The dimensions of the stabilizer and the friction constant c must be selected[1] so as to make the amplitude of rolling as small as possible. Experience shows that a properly designed stabilizer is very effective in damping rolling oscillations of ships. This can be appreciated from Fig. 276, representing the rolling angle ψ of the ship in one of Schlik's experiments. The angle was recorded for two conditions: (1) when the stabilizer was locked and (2) when it was acting.

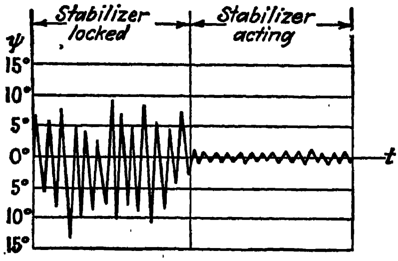

FIG. 276.

Instead of introducing proper friction forces, we can reduce rolling of a ship also by applying to the stabilizer's frame a proper moment with respect to the axis O_1O_1 in Fig. 274. Denoting this moment by M_1 and neglecting the friction forces, we obtain the equations

$$\left.\begin{aligned} J\ddot{\psi} + WL\psi + H\dot{\beta} &= M \sin pt, \\ j\ddot{\beta} + wl\beta - H\dot{\psi} &= M_1. \end{aligned}\right\} \tag{x}$$

Let us now select the variable moment M_1 so as to produce oscillations of the stabilizer's frame in accordance with the equation

$$\beta = \kappa\psi, \tag{y}$$

where κ is a positive number.

Substituting in the first of Eqs. (x), we obtain

$$J\ddot{\psi} + \kappa H\dot{\psi} + WL\psi = M \sin pt. \tag{z}$$

[1] More information regarding the proper design of the stabilizer can be found in Klein and Sommerfeld, *op. cit.*, vol. 4, p. 833.

This is the known equation of damped vibrations, the damping being accomplished by gyroscopic action instead of friction. When ψ is found from Eq. (z), we substitute it and also expression (y) for β into the second of Eqs. (x) and obtain in this way the moment M_1 required to produce the proper motion of the frame. This idea of stabilizing the ship by communicating to the frame of the gyroscope the proper oscillations depending on the rolling angle ψ is used in the Sperry stabilizer. In this device, the motion of the stabilizer's frame is accomplished by a special servomotor, the action of which depends on the rolling motion of the ship. In recent times, this type of stabilizer has found a much wider application than the Schlick's stabilizer described above.

52. Monorail Car Stabilization.—In the case of a monorail car, a cross section of which is shown in Fig. 277, the stability of the system is accomplished by the use of a gyroscopic stabilizer. The stabilizer consists of a gyroscope G the axis OO of which is mounted in a frame NN which is free to rotate about its axis O_1O_1. The center of gravity of the stabilizer is above the axis O_1O_1, so that the position of the frame shown in the figure is unstable if the gyroscope is not in rotation. The car is also in unstable equilibrium, since its center of gravity is above the top of the rail. To explain how the stability of the car can be accomplished by bringing the gyroscope to a high speed of rotation, we begin with a discussion of stability of the gyroscope, the center of gravity of which is above the point of support O (Fig. 278). Assuming that the axis of the gyroscope makes only a small angle with the vertical axis and defining its direction by the angles ψ and β, as before, we can write equations of motion of the gyroscope by using Eqs. (g) of Art. 50, derived for the gyroscopic pendulum. To take care of the fact that the center of gravity of the gyroscope in Fig. 278 is above the fixed point O while in the gyroscopic pendulum of Fig. 272 it is below that point, we have only to substitute $-l$ for l in Eqs. (g). This gives

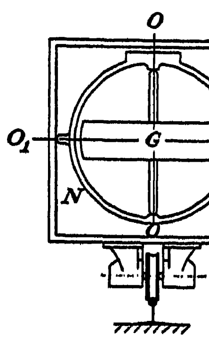

FIG. 277.

$$\left.\begin{aligned} I_1\ddot{\psi} + I\omega\dot{\beta} &= Wl\psi, \\ I_1\ddot{\beta} - I\omega\dot{\psi} &= Wl\beta. \end{aligned}\right\} \qquad (a)$$

Eliminating β from these equations, we obtain

$$I_1^2\,\psi^{\mathrm{IV}} + (I^2\omega^2 - 2I_1Wl)\ddot{\psi} + W^2l^2\psi = 0. \qquad (b)$$

The corresponding characteristic equation is

$$I_1^2m^4 + (I^2\omega^2 - 2I_1Wl)m^2 + W^2l^2 = 0. \qquad (c)$$

It gives

$$m^2 = \frac{-(I^2\omega^2 - 2I_1Wl) \pm \sqrt{(I^2\omega^2 - 2I_1Wl)^2 - 4W^2l^2I_1^2}}{2I_1^2}$$
$$= \frac{-(I^2\omega^2 - 2I_1Wl) \pm I\omega\sqrt{I^2\omega^2 - 4I_1Wl}}{2I_1^2}. \quad (d)$$

The motion of the gyroscope is stable as long as its axis makes a small angle with the vertical x-axis and performs only small oscillations about that axis. This condition is satisfied if all four roots of the characteristic equation (c) are pure imaginary numbers. To prove this, let us assume that there is a real root $m_1 = a$. Since Eq. (c) contains only m^2 and m^4, we conclude that $m_2 = -a$ is also the root of Eq. (c). This shows that

$$\psi = Ce^{at} \quad \text{and} \quad \psi = Ce^{-at}$$

are solutions of Eq. (b). Since either a or $-a$ is a positive number, one of these solutions represents an indefinite growth of the angle ψ with time and indicates that the assumed motion of the gyroscope is unstable. Similarly, if there is a complex root $m_1 = a + bi$, there will also be the root $m_2 = -a - bi$. The conjugate complex numbers $a - bi$ and $-a + bi$ will also be roots of Eq. (c), and the general solution of Eq. (b) is

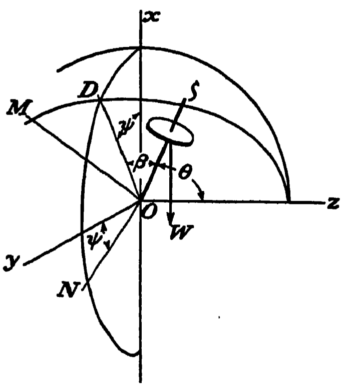

FIG. 278.

$$\psi = e^{at}(C_1\cos bt + C_2\sin bt) + e^{-at}(C_3\cos bt + C_4\sin bt).$$

Since either the factor e^{at} or e^{-at} increases indefinitely with time, we conclude that complex roots of Eq. (c) also indicate that the motion of the gyroscope is unstable. Stability is realized only if all roots of Eq. (c) are imaginary. From expression (d), we see that this occurs when m^2 is negative, *i.e.*, when we have

$$I^2\omega^2 - 4I_1Wl > 0. \quad (e)$$

Assuming that this condition is satisfied and using the notations

$$p_1^2 = \frac{(I^2\omega^2 - 2I_1Wl) - I\omega\sqrt{I^2\omega^2 - 4I_1Wl}}{2I_1^2},$$
$$p_2^2 = \frac{(I^2\omega^2 - 2I_1Wl) + I\omega\sqrt{I^2\omega^2 - 4I_1Wl}}{2I_1^2},$$

we can write the general solution of Eq. (b) in the form

$$\psi = C_1\cos p_1t + C_2\sin p_1t + C_3\cos p_2t + C_4\sin p_2t. \quad (f)$$

Using this solution in Eqs. (*a*), we shall find

$$\beta = D_1\cos p_1t + D_2\sin p_1t + D_3\cos p_2t + D_4\sin p_2t, \qquad (g)$$

where the constants $D_1 \ldots D_4$ are similar to those that appear in Eq. (*m*) of the preceding article. Thus we have altogether four constants of integration $C_1 \ldots C_4$, which in each particular case can be calculated if the initial values ψ_0, β_0 of the angles and the initial velocities $\dot{\psi}_0$, $\dot{\beta}_0$ are given.

We see from this discussion that motion of the gyroscope shown in Fig. 278 can always be made stable; it is necessary only to give to the gyroscope a sufficient speed of rotation ω such that condition (*e*) is satisfied. This property is utilized in stabilization of the monorail car in Fig. 277. The system represented by this car together with the stabilizer is similar to that of the ship and stabilizer in Fig. 274 of the preceding article. The difference lies only in the fact that in the present case, the centers of gravity of the car and of the stabilizer are above the corresponding axes of rotation while, in the case of ship stabilization, they were below those axes. From this consideration, it follows that the equations of motion of the monorail car can be obtained from Eqs. (*e*) or (*f*) of the preceding article by substituting in those equations $-L$ and $-l$ for L and l. In this manner, we obtain, from Eqs. (*f*),

$$\left.\begin{aligned} J\ddot{\psi} - WL\psi + H\dot{\beta} &= 0, \\ j\ddot{\beta} - wl\beta - H\dot{\psi} &= 0. \end{aligned}\right\} \qquad (h)$$

Here ψ is the angle of rotation of the car about the axis taken along the top of the rail; β is the angle of rotation of the stabilizer about the axis O_1O_1, J is the moment of inertia of the car together with the stabilizer about the axis along the top of the rail, W is the weight of the car with the stabilizer, L the distance of their common center of gravity from the top of the rail, $H = I\omega$ is the angular momentum of the gyroscope, j is the moment of inertia of the frame NN together with the gyroscope about the axis O_1O_1, w is their joint weight, and l is the distance of their center of gravity from the axis O_1O_1.

To investigate stability of motion of the monorail car, we eliminate β, as before, from Eqs. (*h*); and in this manner, we obtain

$$Jj\psi^{\text{IV}} - (WLj + wlJ - H^2)\ddot{\psi} + wlWL\psi = 0. \qquad (i)$$

Using the notation

$$k = -(WLj + wlJ - H^2), \qquad (j)$$

we write the characteristic equation

$$Jjm^4 + km^2 + wlWL = 0. \qquad (k)$$

Proceeding as in the case of Eq. (*b*), we conclude that the motion of the monorail car system is stable if all four roots of Eq. (*k*) are imaginary numbers, which requires that m^2 be a real negative number. Solving Eq. (*k*) for m^2, we obtain

$$m^2 = \frac{-k \pm \sqrt{k^2 - 4JjwlWL}}{2Jj}. \tag{l}$$

This quantity will be a negative real number if

$$k > 0 \qquad \text{and} \qquad k^2 - 4JjwlWL > 0. \tag{m}$$

We see that for stability of the car it is essential to have the center of gravity of the stabilizer above the axis of rotation O_1O_1 in Fig. 277. If the center of gravity were below the axis O_1O_1, we should have to change the sign of l; then the second term under the radical in expression (*l*) becomes positive, and one of the two roots for m^2 becomes positive also, which indicates instability of motion of the car.

The conditions of stability (*m*) will be satisfied if

$$k > 2\sqrt{JjwlWL}$$

or, using notation (*j*),

$$H^2 > (\sqrt{WLj} + \sqrt{wlJ})^2. \tag{n}$$

Since $H = I\omega$ is the angular momentum of the gyroscope, we always can select a sufficiently high angular speed ω to satisfy condition (*n*) and make the monorail car motion stable.

When ω is very large, we can assume

$$\sqrt{k^2 - 4JjwlWL} \approx k\left(1 - \frac{2JjwlWL}{k^2}\right) \approx k - \frac{2JjwlWL}{H^2},$$

and Eq. (*l*) gives

$$m_1^2 = -\frac{wlWL}{H^2}, \qquad m_2^2 = -\frac{H^2}{Jj}.$$

Using the notations

$$p_1^2 = -m_1^2 \qquad \text{and} \qquad p_2^2 = -m_2^2,$$

we then represent the general solution of Eq. (*i*) in the form

$$\psi = C_1\cos p_1t + C_2\sin p_1t + C_3\cos p_2t + C_4\sin p_2t.$$

The expression for β is obtained from Eqs. (*h*), which give

$$\beta = D_1\cos p_1t + D_2\sin p_1t + D_3\cos p_2t + D_4\sin p_2t,$$

where the constants $D_1 \ldots D_4$ are expressed through the constants $C_1 \ldots C_4$ by expressions similar to those at the bottom of page 369.

We see that the motions of the car and of the stabilizer are obtained

by superposition of two small oscillations of the frequencies

$$p_1 \approx \frac{1}{H}\sqrt{wlWL}, \qquad p_2 \approx H\sqrt{\frac{1}{Jj}}.$$

For a large H, the first frequency is small and the second is large. We see that on slow oscillations of the car and the stabilizer, high-frequency vibrations are superimposed.

In the preceding discussion, damping was neglected. To derive equations of motion with damping, we start with Eqs. (e) of the preceding article. Substituting $-l$ and $-L$ for l and L, we obtain the necessary equations from which an equation of fourth order for ψ can be derived.

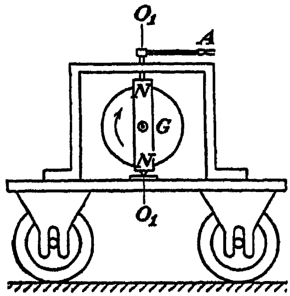

FIG. 279.

From the kind of roots of the corresponding characteristic equation, the conditions required for stability of motion can be established.

The above-discussed scheme of a gyroscopic stabilizer was used in patents of A. Scherl[1] and P. Schilowsky, who worked simultaneously and independently on this problem. Another scheme was proposed by L. Brennan, who used a gyroscope with horizontal axis perpendicular to the direction of the track, as shown in Fig. 279. It may be seen from this figure that by proper movements of the handle A producing rotation of the frame NN in which the gyroscope G is mounted, a moment in the plane perpendicular to the longitudinal axis of the car may be produced. This moment can be used to balance the car. In the actual patent, this balancing is accomplished automatically.

[1] Regarding A. Scherl patent, see paper by Von Barkhausen, *Z. Ver. deut. Ing.*, vol. 54, p. 1738, 1910.

APPENDIX

DIMENSIONAL ANALYSIS AND THEORY OF MODELS

1. Homogeneity and Dimensional Analysis.—In mechanics, we have to deal with *three fundamental units*, namely: the unit of *length*, denoted by L; the unit of *time*, denoted by T; and the unit of *mass*, denoted by M. All other quantities such as acceleration, force, velocity, momentum, energy, etc., can be expressed in terms of these three fundamental units. Thus, for example, acceleration is expressed in units of length and time:

$$\mathrm{a} = \frac{\mathrm{L}}{\mathrm{T}^2}. \tag{a}$$

Similarly, with mass as a *fundamental unit*, force becomes a *derived unit* and is obtained as the product of mass times acceleration; *i.e.*,

$$\mathrm{F} = \frac{\mathrm{ML}}{\mathrm{T}^2}. \tag{b}$$

Energy or work, which is the product of force times length, likewise becomes

$$\mathrm{V} = \frac{\mathrm{ML}^2}{\mathrm{T}^2}. \tag{c}$$

Equations like (*a*), (*b*), and (*c*), giving the dimensions of derived units in terms of the fundamental units are called *dimensional equations.* They can be used in checking the dimensional character of equations derived in the solution of various mechanical problems. All terms of such equations must have identical dimensional form. Take, for example, the expression for displacement s of a particle falling vertically under the action of gravity:

$$s = s_0 + v_0t + \tfrac{1}{2}gt^2. \tag{d}$$

Using the dimensional equations for velocity and acceleration, it can be readily shown that each term in Eq. (*d*) has the dimension of length and the requirement of identity of dimensions of all terms is satisfied. This requirement is sometimes called the *principle of homogeneity of dimensions.* It can be very useful in a preliminary analysis of a mechanical problem in establishing the proper relations among the various quantities involved.

As an example, let us consider the motion of a simple pendulum and assume that it is required to establish a formula for the period τ of

oscillation. We start out by assuming that this period depends on the mass m of the pendulum, on its length l. and on the gravity force defined by the acceleration g. Then

$$\tau = f(m,l,g), \tag{e}$$

where f stands for a function of the quantities m, l, and g. Our problem is to determine the proper function. Since the quantities m, g, and l have different dimensions and cannot be added or subtracted, we assume a function in the form of a product of powers of these quantities and write

$$\tau = Cm^x l^y g^z. \tag{f}$$

Since the left-hand side of this equation has the dimension of time, the right-hand side must have the same dimension. Putting the equation in dimensional form, we obtain

$$\mathrm{T} = \mathrm{M}^x\mathrm{L}^y\left(\frac{\mathrm{L}}{\mathrm{T}^2}\right)^z = \mathrm{M}^x\mathrm{L}^{y+z}\mathrm{T}^{-2z}.$$

Then to satisfy the requirement of homogeneity, we have to take

$$x = 0, \qquad y + z = 0, \qquad 2z = -1. \tag{g}$$

From these equations, we find

$$x = 0, \qquad y = \tfrac{1}{2}, \qquad z = -\tfrac{1}{2},$$

and Eq. (f) becomes

$$\tau = C\sqrt{\frac{l}{g}}. \tag{h}$$

Thus, we obtain the correct form of the expression for the period of oscillation by using the principle of homogeneity of dimensions.

It should be noted that by using this method, we establish only the general form of Eq. (h), but the numerical value of the factor C remains indefinite. To determine it, we must either solve the differential equation of motion of the pendulum or resort to experiment.

It was assumed in the above discussion that the period τ depends only on the quantities m, l, and g, but we know that this is approximately correct only for small angles of oscillation. If the angle is not small, the period depends, to some extent, on the arc of swing s. Taking this into account, we write, instead of Eq. (f),

$$\tau = Cm^x l^y g^z s^u. \tag{i}$$

Then from the principle of homogeneity, we obtain

$$x = 0, \qquad y + z + u = 0, \qquad 2z = -1, \tag{j}$$

from which

$$x = 0, \qquad y = \tfrac{1}{2} - u, \qquad z = -\tfrac{1}{2},$$

and Eq. (i) becomes

$$\tau = C\sqrt{\frac{l}{g}}\left(\frac{s}{l}\right)^u. \tag{k}$$

Again, by using the method of dimensional analysis, we obtain the general formula (k) for the period τ, but the number of Eqs. (j) is insufficient to determine all of the unknowns, and the exponent u remains indefinite. The ratio s/l is a pure number, and the principle of homogeneity imposes no restriction on the power to which it is raised. We can take various values for u without disturbing the homogeneity of Eq. (k) and can write it in the more general form

$$\tau = C\sqrt{\frac{l}{g}}\,\phi\left(\frac{s}{l}\right). \tag{k'}$$

The method of dimensions does not define the function ϕ; but since it contains only the ratio s/l, we conclude that it is not the arc s but the angle of swing that affects the period of oscillation. Having established the general formula (k'), the proper function ϕ can be found by experiment.

As a second example of application of the method of dimensional analysis, let us consider the periods of lateral vibrations of geometrically similar beams. In such case, we assume that the periods of vibration depend on linear dimensions l, modulus of elasticity E, and mass density ρ of the material. Then, proceeding as before, we write

$$\tau = Cl^xE^y\rho^z. \tag{l}$$

Observing that modulus of elasticity E has the dimension of force ÷ area while density ρ is mass ÷ volume, we obtain

$$\mathrm{T} = \mathrm{L}^x\left(\frac{\mathrm{M}}{\mathrm{LT}^2}\right)^y\left(\frac{\mathrm{M}}{\mathrm{L}^3}\right)^z = \mathrm{L}^{x-y-3z}\mathrm{M}^{y+z}\mathrm{T}^{-2y}.$$

The requirement of homogeneity now gives

$$x - y - 3z = 0, \qquad y + z = 0, \qquad 2y = -1,$$

and we obtain

$$y = -\tfrac{1}{2}, \qquad z = \tfrac{1}{2}, \qquad x = 1.$$

Substituting into Eq. (l), we find

$$\tau = Cl\sqrt{\frac{\rho}{E}}. \tag{m}$$

We see that the period of vibration is proportional to the linear dimensions of the beam and to the square root of the ratio of mass density to modulus of elasticity. Thus the method of dimensional analysis again furnishes the general formula (m) for period τ. The numerical value of the factor C, which is easy to calculate in the case of prismatic beams, can be determined experimentally in more complicated cases.

Our assumption that the period of vibration τ depends on linear dimensions, modulus of elasticity, and mass density can be applied in considering vibrations of elastic bodies of any shape. Then using formula (m) and comparing periods of vibration of two geometrically similar bodies of the same material, we conclude that these periods are in the same ratio as the linear dimensions of the bodies. This conclusion holds also in the case of gases and it has been established experimentally that when masses of air are contained in two geometrically similar vessels, the number of vibrations in a given time is inversely proportional to the linear dimensions of the vessel.[1]

As another example, let us consider the time τ required for a planet to make a complete trip around the sun. We assume that this period of the planet will depend on its distance r from the sun, which can be considered as constant; on the mass m of the planet; and on the force of attraction. Then, using Newton's universal law of gravitation,

$$\tau = Cr^x \left(\frac{km}{r^2}\right)^y m^z. \tag{n}$$

Since the quantity in the parentheses of this expression has the dimension of force, the constant k evidently has the dimension of acceleration times length^2, and we write the dimensional equation

$$\mathrm{T} = \mathrm{L}^x \left(\frac{\mathrm{ML}}{\mathrm{T}^2}\right)^y \mathrm{M}^z.$$

Applying the principle of homogeneity to this equation, we obtain

$$x = \tfrac{1}{2}, \qquad y = -\tfrac{1}{2}, \qquad z = \tfrac{1}{2},$$

and Eq. (n) becomes

$$\tau = Cr^{\frac{3}{2}}k^{-\frac{1}{2}}.$$

From this equation, we conclude that

$$\frac{\tau^2}{r^3} = \text{const.},$$

which represents Kepler's third law of planetary motion (see page 92).

[1] This is known as Savart's theorem. See *Ann. chim.*, Paris, 1825.

From this and the preceding example, we see that useful conclusions regarding mechanical problems can sometimes be reached by dimensional analysis alone. Naturally, however, we must have a correct idea of the factors that influence a given physical phenomenon before we can apply the method.

Dimensional analysis is especially useful in dealing with complicated problems that cannot be solved analytically but require empirical development based on experiment. In such cases, the general form of the required equation is established by dimensional analysis and then experiments are made to determine the numerical values of constants that may appear in the equation. As an example, let us derive an equation for the axial thrust P produced by a propeller of given shape rotating in air about a fixed axis with constant angular velocity ω. For the total air pressure dP acting on an element ds of the surface of a blade, we formulate the expression[1]

$$dP = kv^2 ds\, \phi(\alpha), \tag{o}$$

in which k is a constant, v is the velocity of the element under consideration, and $\phi(\alpha)$ is an unknown function of the angle α between the velocity v and the surface of the element. Since the shape of the propeller is given, all quantities except k in Eq. (o) can be expressed in terms of the diameter d of the propeller and its angular velocity ω. Thus the total thrust P can be represented by the formula

$$P = f(k,d,\omega), \tag{p}$$

in which the constant k, as can be seen from Eq. (o), has the dimension FT^2/L^4. To satisfy the condition of homogeneity in Eq. (p), we have to take the function f linear and also take d to the fourth power and ω to the square, which finally gives

$$P = Ckd^4\omega^2. \tag{q}$$

By making tests with a propeller of given size and running at a given speed, we can determine the constant Ck experimentally and then use Eq. (q) to compute the thrust for any other geometrically similar propeller running at any other speed. We see that dimensional analysis used as a preliminary to experimental investigation of a given problem can be very helpful.

Resistance to fluid motion in a pipe can also be investigated by using the method of dimensional analysis. Experiments show that in the case of steady flow of an incompressible fluid through a smooth straight pipe, the resistance τ per unit of surface area of pipe depends on the viscosity μ

[1] We neglect here the effect of roughness of the propeller surface and also the effect of compressibility of the air.

of the fluid, on its mass density ρ, on the diameter d of the pipe, and on the average velocity v of flow.[1] Then an equation for τ can be written as follows:

$$\tau = C\mu^x\rho^y d^z v^u. \tag{r}$$

Observing that τ has the dimension FL^{-2} and that viscosity has the dimension FTL^{-2}, the dimensional equation becomes

$$\frac{M}{LT^2} = \left(\frac{M}{LT}\right)^x\left(\frac{M}{L^3}\right)^y L^z\left(\frac{L}{T}\right)^u.$$

The requirement of homogeneity then gives

$$x + y = 1, \qquad -x - 3y + z + u = -1, \qquad -x - u = -2.$$

We see that these three equations are insufficient to determine all four unknowns and one of them must remain indefinite. Let this be u; then

$$x = 2 - u, \qquad y = u - 1, \qquad z = u - 2,$$

and Eq. (r) becomes

$$\tau = C\frac{\mu^2}{\rho d^2}\left(\frac{\rho v d}{\mu}\right)^u. \tag{s}$$

It can be readily shown that the expression in the parentheses, like s/l in Eq. (k), is a pure number. Hence the principle of homogeneity imposes no restriction on the exponent u. Taking, for example, $u = 1$, we obtain

$$\tau = C\frac{\mu v}{d},$$

which is the well-known Hagen-Poiseuille law for laminar flow in a straight pipe.

In the more general case when the flow is turbulent, Eq. (s) can be generalized by taking a series of values for u. Then combining terms, we write the equation in the form

$$\tau = C\frac{\mu^2}{\rho d^2}\,\phi\left(\frac{\rho v d}{\mu}\right), \tag{s'}$$

where ϕ stands for any function of $\rho vd/\mu$. We can also multiply the right-hand side of this equation by the pure number $(\rho vd/\mu)^2$ and obtain for τ the formula

$$\tau = C\frac{\mu^2}{\rho d^2}\left(\frac{\rho v d}{\mu}\right)^2\phi\left(\frac{\rho v d}{\mu}\right) = Cv^2\rho\,\psi\left(\frac{\rho v d}{\mu}\right)$$

[1] The average velocity v is defined as the quantity of flow per unit time divided by the cross-sectional area of the pipe.

or, finally,

$$\frac{\tau}{\rho v^2} = C\psi\left(\frac{\rho v d}{\mu}\right). \qquad (s'')$$

The equation in this form has proved very useful in experimental studies of flow of fluid in smooth pipes. By plotting $\tau/\rho v^2$ against $\rho v d/\mu$, we obtain a curve that applies to the flow of any kind of fluid and for any diameter of the pipe. Thus the results of experiments with the flow of water in small pipes can be applied with confidence to the flow of oil in large pipes, etc. The pure number $\rho v d/\mu$ was first introduced by Osborne Reynolds and is called *Reynolds number*.[1]

It will be noted that in the preceding examples where our equations involved only three independent variables, we were able, in each case, to evaluate all of the exponents by the principle of homogeneity, while in those equations such as (i) and (r), which contained four independent variables, one of the exponents remained indefinite. In each of these latter cases, however, we found that the quantities containing the indefinite exponent u combined so as to give a pure number to this power. Thus, in dealing with the simple pendulum, we found that the ratio s/l defining the angle of swing was significant, while in discussing the resistance to flow of fluids in smooth pipes, we found the Reynolds number $\rho v d/\mu$.

In general, it can be proved that any correct and complete equation of the form

$$\Phi(Q_1, Q_2 \ . \ . \ . \ Q_n) = 0, \qquad (1)$$

involving n physical quantities Q all expressible in terms of k fundamental units, can be reduced to the form

$$\Psi(\Pi_1, \Pi_2 \ . \ . \ . \ \Pi_{n-k}) = 0, \qquad (2)$$

where Π_1, Π_2 . . . Π_{n-k} are all the independent dimensionless products that can be made up by combining in any way the n physical quantities in Eq. (1). This is known as Buckingham's pi-theorem, which is now widely used in dimensional analysis.[2]

To show the application of this theorem, we reconsider the problem of resistance to flow of fluids through smooth pipes as discussed above. Having decided on the physical quantities involved, we start with the equation

$$F_1(\tau, \rho, v, d, \mu) = 0, \qquad (t)$$

[1] O. Reynolds, *Phil. Trans. Roy. Soc.*, 1883, pp. 174, 935.

[2] See Edgar Buckingham, "Model experiments and the form of empirical equations," *Trans. Am. Soc. Mech. Engrs.*, vol. 37, pp. 263–296, 1915. The proof of the theorem is given in the appendix to this paper. The terms pi-theorem comes from the use of the Greek letter Π signifying product.

analogous to Eq. (r). Writing down the dimensions of the five quantities

$$\tau = \frac{M}{LT^2}, \qquad \rho = \frac{M}{L^3}, \qquad v = \frac{L}{T}, \qquad d = L, \qquad \mu = \frac{M}{LT},$$

and remembering that we have $k = 3$, we find by trial[1] that two possible independent dimensionless products in this case are[2]

$$\Pi_1 = \frac{\tau}{\rho v^2}, \qquad \Pi_2 = \frac{\rho v d}{\mu},$$

and Eq. (2) becomes

$$\Psi\left(\frac{\tau}{\rho v^2}, \frac{\rho v d}{\mu}\right) = 0,$$

or

$$\frac{\tau}{\rho v^2} = F\left(\frac{\rho v d}{\mu}\right),$$

as obtained before [see Eq. (s'')].

The use of the pi-theorem has some advantage if we have to deal with a large number of physical quantities, since it reduces the number of variables by k. In general, however, we cannot accomplish anything by its use that could not be done by the more direct method used before. In this respect, the theorem is somewhat analogous to the Lagrangian method of writing equations of motion. It does not so much promote insight into dimensional analysis as reduce the problem to a systematic procedure requiring the minimum of mental effort.

2. Theory of Models.—It is common practice in engineering to use models in solving complicated mechanical problems. Models of such structures as bridges and dams are sometimes used to investigate the strength of those structures under actual conditions. Likewise models of ships are tested under various speeds to establish the necessary power of engines. Models are also widely used in airplane design.

In the case of statical problems, the relations between the mechanical properties of the model and of the prototype are comparatively simple. For example, if we are interested in stresses produced in a structure by live load and use the known formulas of strength of materials, we find that the stresses are proportional to the applied forces and inversely proportional to the squares of linear dimensions. This indicates that if the model is geometrically similar to the prototype with the ratio $1/\lambda$ of linear dimensions to corresponding actual dimensions, then the

[1] If the number of quantities n is large, there are definite rules of procedure for finding the $n - k$ independent products. See Buckingham, *loc. cit.*

[2] Any two independent products may be taken. We choose these two in order to come directly to an equation of the form (s'').

stresses in the model will be the same as in the actual structure provided the forces applied to the model are in the ratio $1/\lambda^2$ to the actual forces. If the materials of model and prototype are the same, equal stresses produce equal strains and the deformed shape of the model will be geometrically similar to that of the prototype. Multiplying the deflections measured on the model by λ, we obtain the actual deflections of the structure.

If we are interested in elastic stability of compressed members of a structure, we observe that the magnitude of the critical stress, given by Euler's formula, is

$$\sigma_{cr} = \frac{\pi^2 E r^2}{l^2} \tag{a}$$

This stress is independent of linear dimensions and, for materials with equal moduli E, is the same for model and prototype. Equations similar to Eq. (*a*) can be established in other cases of instability, which indicates that from model tests, we can find the load at which lateral buckling of members of the actual structure may occur.

If we wish to find from model tests the stresses produced in the prototype by dead load, such as the proper weight of a structure, we find that this problem is more complicated. When the linear dimensions are reduced in the ratio $1/\lambda$, the weight is reduced in the ratio $1/\lambda^3$, which indicates that the stresses produced in the model by proper weight will be λ times smaller than in the prototype if the materials are the same. To have the same stresses in the model as in the prototype, we must use for the model a material the density of which is λ times larger than that in the actual structure. This requirement is difficult to realize; and in practical model tests, the insufficiency of proper weight is usually compensated by dead loads distributed over the model.

If we are investigating the behavior of a structure only within the elastic limit of the material, we do not need to compensate for insufficiency of dead load and can dispense with the requirement of equal stresses in model and prototype. It is only necessary to measure the dead load stresses in the model and then multiply by λ to obtain the actual stresses in the structure. In such cases, however, the dead load stresses and deformations in the model will usually be very small and will require very sensitive measuring instruments. To remove this difficulty, the use of model materials having a small modulus of elasticity has been recommended. For example, in studying stresses in arch dams, models made of rubber have been used.[1]

[1] See paper by A. V. Karpov and R. L. Templin, *Proc. Am. Soc. Civil Engrs.*, April, 1934.

To investigate the behavior of a structure beyond the elastic limit and establish its ultimate strength under dead load, we have to use for the model the same material as for the actual structure. To have stresses of the same magnitude as in the prototype, the model must then be put in a force field, the intensity of which is λ times larger than that of the gravitational field. This can be accomplished by revolving the model in a centrifuge.[1] The centrifugal force field can be assumed as approximately uniform if the radius of the centrifuge is large in comparison with the dimensions of the model. Such experiments have proved very useful in solving problems encountered in mine structures and in the construction of tunnels. Failure of such structures may occur due to dead load alone; and since the material does not follow Hooke's law, a theoretical solution becomes very involved. In such cases, a reliable value of the ultimate strength can be obtained from model tests alone.

If we go from statics to dynamics, the relations between mechanical quantities pertaining to the model and to the prototype become more involved. We have to consider not only the ratio λ of linear dimensions of the two geometrically similar systems but also the ratio of their masses and of their velocities. Take, for example, a geometrically similar model of an engine, and assume that we use for this model the same material, the same angular velocities, and the same gas pressure as for the prototype. Then the total forces on the two pistons will be in the ratio $1/\lambda^2$. The weights of the corresponding parts will be in the same ratio as their volumes; *i.e.*, in the ratio $1/\lambda^3$. Velocities and accelerations of corresponding points will be in the ratio $1/\lambda$. The inertia forces of corresponding parts, having the dimension of mass times acceleration, will be in the ratio $1/\lambda^4$. From this we conclude that stresses produced by gas pressure in the model and in the prototype will be the same, stresses produced by gravity forces will be in the ratio $1/\lambda$, and stresses due to inertia forces will be in the ratio $1/\lambda^2$. It is seen that in this case we do not have complete similarity. Nevertheless, under certain conditions, useful information can be obtained from model tests. If the engine is not very large and runs at slow speed, the stresses are produced principally by gas pressure, and we obtain these with good accuracy from model tests. In another extreme case, when the engine is running at high speed, the stresses produced by inertia forces predominate, and we can obtain these from model tests if the angular velocities of the model and of the prototype are taken in the ratio $\lambda/1$. Then linear velocities of the corresponding points of the model and of the prototype are the same, accelerations are in the ratio $\lambda/1$, and inertia forces are in the ratio $1/\lambda^2$.

[1] Experiments of this kind were made by P. B. Bucky; see his paper presented at the semiannual meeting of the American Society of Mechanical Engineers, June, 1932.

Such conditions are satisfied, for example, in model tests of turbine disks and flywheels if the peripheral velocities of model and prototype are equal.

For a more complete study of dynamical models, let us consider two geometrically similar systems, the linear dimensions of which are in the ratio λ, and assume that the masses of the corresponding portions of the systems are in the ratio μ. We say that these two systems perform similar motions if they pass through geometrically similar configurations in intervals of time that are in a constant ratio τ. Thus two geometrically similar pendulums perform similar motions if the initial angles of inclination are taken equal and the pendulums are released without initial velocity. Two geometrically similar engines running each with its constant angular velocity also perform similar motions. If equations of motion for one of these systems are derived, they can be used also for the second system; we have only to substitute λl, instead of length l, μm instead of mass m, and τt, instead of time t. Velocities of the corresponding points for similar configurations will be in the ratio $\lambda\tau^{-1}$, and accelerations will be in the ratio $\lambda\tau^{-2}$. The forces acting on corresponding portions of the two systems must be in the same ratio as the products of mass times acceleration. Denoting this ratio by φ, we obtain

$$\varphi = \frac{\mu\lambda}{\tau^2} \tag{3}$$

This relation establishes the relation among the four constants λ, μ, τ, and φ and is of great importance in the theory of models. It states that only three of the quantities can be taken arbitrarily; the fourth is then determined by Eq. (3).[1]

Let us consider again the model of an engine and establish under what condition the model and the prototype will perform similar motions. If model and engine are of the same material, the gravity forces of corresponding portions will be in the ratio $\mu = \lambda^3/1$. For similarity of motion, it is necessary that all other forces be in the same ratio. For example, the total pressures on the pistons must be in the ratio $\lambda^3/1$. This can be accomplished by using for the model a gas pressure λ times higher than that for the prototype. From Eq. (3), we see that to make inertia forces also in the ratio $\lambda^3/1$, we must take

$$\lambda = \tau^2. \tag{b}$$

Observing that $\lambda/\tau = v$ is the ratio of the velocity of any point of the engine to the velocity of the corresponding point of the model, we con-

[1] The principle of similitude in dynamics was established by Newton in Prop. 32, Sec. VII, of the second book of the "Principia."

clude, from Eq. (*b*), that for similarity of motions, we must have

$$\nu^2 = \lambda; \tag{c}$$

i.e., the squares of linear velocities must be in the same ratio as linear dimensions. If we take into account friction forces in bearings and assume them proportional to the reactions transmitted through the bearings, we shall find that friction forces are also in the ratio $\lambda^3/1$. We obtain the same conclusion also for air resistance if we assume this resistance proportional to area and to the square of velocity. It is seen that in the case of a model of an engine, all requirements regarding similarity of motion can be satisfied and we can arrange the model so that its motion will be similar to that of the engine. However, this will be only dynamical similarity; from motion of the model, we can draw conclusions about the motion of the engine, but not about the strength of machine parts. Since all forces are in the ratio $\lambda^3/1$, the ratio of stresses in the engine to those in the model will be $\lambda/1$; and with respect to mechanical strength, the engine will be weaker than the model.

To have the same stresses in model and engine, the forces must be in the ratio $\lambda^2/1$. Theoretically, this can be accomplished by taking for the model the same gas pressure as for the engine and using a material the density of which is λ times larger than that used in the engine. Then the gravity forces of corresponding parts will be in the ratio $\lambda^2/1$; and if we fulfill condition (*b*), the motion of the model will be similar to that of the engine,[1] and in both cases we shall have the same stresses. But now since the materials are different, it will be impossible to make conclusions from model tests regarding the mechanical strength of the engine.

As a second example, let us consider the use of models in ship design. Since these models are used to find the water resistance at various speeds of the ship, they do not need to be similar to the ship in all respects. They must have only the immersed portions of their outer surfaces geometrically similar; and if the linear dimensions are in the ratio $1/\lambda$, the weight will be in the ratio $1/\lambda^3$. The moving model produces on the water surface a complicated wave motion, which is the principal source of resistance (see page 26). Assuming that the motion of water around the model is similar to that around the ship and that condition (*c*) is fulfilled, we conclude from Eq. (3) that the ratio of resistance to motion of the actual ship to the measured resistance of the model is equal to $\mu = \lambda^3$. If we, take, for example, $\lambda = 9$, *i.e.*, the linear dimensions of the model are one-ninth of the corresponding dimensions of the ship, we obtain from Eq. (*c*) $\nu = 3$, which states that the velocity of the model should be one-

[1] We neglect here friction forces and air resistance.

third that of the ship. Then the resistance to propulsion of the model will be $1/\lambda^3 = \frac{1}{729}$ of the resistance of the ship. When the velocity of the ship and, from the model test, the magnitude of the resistance to be expected are known, the proper power of the engines can be selected.[1]

As a last example, let us consider model tests used to investigate the dynamical deflection of a bridge under the action of live load. As a basis of discussion, we take a problem proposed by Routh.[2]

Experiments are to be made on the deflection of a bridge 50 ft. long and weighing 100 tons, when an engine weighing 20 tons passes with a velocity of 40 m.p.h., by means of a model bridge 5 ft. long and weighing 100 oz. Find the weight of the model engine; and if the model bridge be of such stiffness that its statical central deflection under the model engine be one-tenth of the statical central deflection of the bridge due to the engine, show that the velocity of the model must be 40 m.p.h./$\sqrt{10}$.

To obtain dynamical similarity, we select the weight of the model engine in the same ratio to the weight of the model bridge as that between live and dead load in the actual bridge. Since the rigidity of the model bridge is such that the ratio of statical deflections of the model to the corresponding deflections of the bridge is the same as the span ratio, we conclude that for very slow motion, when we can assume static conditions with good accuracy, the deflection curve of the model will be geometrically similar to that of the actual bridge. Assume now that velocities are not small so that dynamical conditions must be considered. The ratio of the span of the bridge to that of the model is $\lambda = 10$. The corresponding masses are in the ratio $\mu = 100$ ton/100 oz. The gravity forces are in the ratio $\varphi = \mu = 100$ ton/100 oz. For geometrically similar deflection curves, elastic reactions will also be in this ratio. Substituting these ratios into Eq. (3), we conclude that for dynamical similarity, it is necessary to have

$$\tau = \sqrt{\lambda},$$

which indicates that the ratio of the velocity of the model engine to the velocity of the actual engine must be

[1] The methods of determining the resistance to the motion of ships from model tests was developed by W. Froude. He observed also that the resistance determined on the basis of dynamical similitude does not give the complete resistance and that friction forces depending on the magnitude and roughness of the immersed surface of the ship must be considered. See *Brit. Assoc. Rept.*, 1872, p. 118; 1874, p. 249. A modern discussion of this question can be found in the article by F. Horn in "Handbuch der Physikalischen und Technishen Mechanik," Vol. 5, p. 552.

[2] See Routh's "Dynamics of a System of Rigid Bodies," 6th ed., vol. I, p. 294, 1897.

$$1 : \frac{\lambda}{\tau} = \sqrt{\lambda} = \sqrt{10}$$

as is stated above. If this requirement is fulfilled, bridge and model will perform similar motions and, dynamical deflections of the two will be in the ratio $1/\lambda$. It should be noted that in this case where we are interested only in deflections, the model bridge does not need to be geometrically similar to the actual bridge. Its performance is completely defined by its length, its weight, and its flexural rigidity.[1]

A theoretical solution of the problem (see page 311) indicates that the ratio of dynamical to statical deflections is approximately equal to $1/(1 - \alpha)$ where α represents the ratio of the period of the fundamental type of vibration to twice the time l/v required for the load to pass over the bridge. If we keep α for the model the same as for the bridge, the ratio between the corresponding dynamical deflections will be the same as between the statical deflections and will be equal to $1/\lambda$. To calculate α, we must know the period T of natural vibration of a beam. Using, for this purpose, Eq. (m) of the preceding article, we obtain

$$T = Cl\sqrt{\frac{m}{l^3E}} = C\sqrt{\frac{ml^3}{l^4E}} = C_1\sqrt{\frac{Wl^3}{EIg}}$$

which indicates that the period of vibration is proportional to the square root of the statical deflection. In our case, statical deflections of the bridge and model are in the ratio $\lambda/1$. Hence their periods of vibrations will be in the ratio $\sqrt{\lambda}/1$. In the same ratio also are the times l/v required for the loads to pass over the spans. This indicates that α is the same for the bridge as for the model and the model can be used for determining dynamical deflections of the bridge.[2]

[1] The flexural rigidity is assumed uniform along the span.

[2] Several other examples of the use of models in the investigation of vibration problems are described in the paper by J. P. Den Hartog, *Trans. Am. Soc. Mech. Engrs.*, vol. 54, p. 153, 1932.

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